Laplacian State Transfer on Graphs

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- The transfer of quantum states is a crucial operation in quantum communication and quantum computation, as it enables the exchange of quantum information between different quantum processing units.

- The problem of state transfer is setting up a collection of qubits that allow the information to be transferred from one location to another by using continuous-time quantum walks.

- Continuous-time quantum walks, initially used by Farhi and Gutmann [7], plays an important role in analysing various quantum transportation phenomena.

Definition

- Let G be a finite, simple and undirected graph with adjacency matrix A.
- The Laplacian matrix of G is defined as $L = D - A$, where D is the degree matrix of G.
- The transition matrix

$$U_L(t) := e^{itL} = \sum_{n \geq 0} \frac{(itL)^n}{n!},\ i = \sqrt{-1},\ t \in \mathbb{R}$$

  determines a continuous-time quantum walk on G.
- Since L is real symmetric, $U_L(t)$ is symmetric and also unitary.

Spectral Decomposition

- Suppose the Laplacian matrix L of G has distinct eigenvalues

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_d.$$

- The spectral decomposition of L is

$$L = \sum_{r=0}^{d} \lambda_r F_r,$$

where F_r is the orthogonal projection of λ_r. Furthermore,

- $\sum_{r=0}^{d} F_r = I.$
- $F_r^2 = F_r$ and $F_r F_s = 0$ if $r \neq s$.
- If $f(x)$ is a polynomial function, then $f(L) = \sum_{r=0}^{d} f(\lambda_r) F_r.$

Laplacian Perfect State Transfer

- A graph G is said to have Laplacian perfect state transfer (LPST) at time $\tau \in \mathbb{R}$, if
$$U_L(\tau)\mathbf{e}_u = \gamma \mathbf{e}_v, \ \gamma \in \mathbb{C}. \tag{1}$$
- In case $u = v$ and $\tau \neq 0$, the graph G is periodic at vertex u.
- The study of perfect state transfer in quantum spin networks was initiated by Bose [4] in the context of information transfer.
- By spectral decomposition, the transition matrix
$$U_L(t) = \sum_{r=0}^{d} e^{it\lambda_r} F_r. \tag{2}$$
- The eigenvalue support of a vertex u is the set,
$$\Lambda_u(G) = \{\lambda_j : F_j e_u \neq 0\}.$$

Some known results

- In [6], we see that for any graph G if $u \in V(G)$ is periodic, then the eigenvalues in the eigenvalue support of vertex u must be integers.
- In the study of state transfer, these quantum walks relative to adjacency and Laplacian matrix of a regular graph are equivalent.

Some known results

- In [6], we see that for any graph G if $u \in V(G)$ is periodic, then the eigenvalues in the eigenvalue support of vertex u must be integers.
- In the study of state transfer, these quantum walks relative to adjacency and Laplacian matrix of a regular graph are equivalent.
- Bose et al. [4] showed that the complete graph K_n does not exhibit LPST, however, deleting an edge in K_{4n} allows LPST between the vertices of removed edge.
- Coutinho and Liu [5] showed that the only path that allows LPST is P_2.
- The small graphs which exhibit LPST are $P_2,\ C_4$ and $\ K_4\backslash e$.

Laplacian Pretty Good State Transfer

Laplacian perfect state transfer is a very rare phenomena, so a relaxation is considered.

- A graph G is said to have Laplacian pretty good state transfer (LPGST) if there is a sequence $\{\tau_k\} \in \mathbb{R}$ such that

$$\lim_{k\to\infty} U_L(\tau_k)\mathbf{e}_u = \gamma\mathbf{e}_v,\ \gamma \in \mathbb{C}. \tag{3}$$

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$$\lim_{k\to\infty} U_L(\tau_k)\mathbf{e}_u = \gamma\mathbf{e}_v,\ \gamma \in \mathbb{C}. \tag{3}$$

- In case $u = v$, the graph G is almost periodic at u.

- In [2], we see that LPGST occurs on path P_n between the extremal vertices if and only if n is a power of 2.

Corona of graphs

The corona product $G \circ H$ of two graphs G and H (where G has n vertices and m edges) is defined as the graph G' obtained by taking one copy of G and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H by an edge.

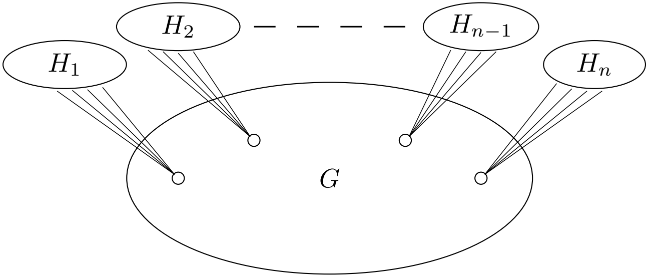

Figure: Corona $G \circ H$.

Theorem [3]

Let G be a graph on n vertices and H be a graph on m vertices. suppose G has spectrum $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_p$ with multiplicities $r_0, r_1, \ldots, r_p$, and H has spectrum $0 = \mu_0 < \mu_1 < \ldots < \mu_q$ with multiplicities ${r_0}', {r_1}', \ldots, {r_q}'$. Then the corona $G \circ H$ has the following spectrum:

- 1 with multiplicity $n({r_0}' - 1)$;
- $\mu_j + 1$ with multiplicity $n{r_j}'$, for each $j = 1, 2, \ldots, q$;
- $\frac{1}{2}(\lambda_j + m + 1 \pm \sqrt{(m + \lambda_j - 1)^2 + 4m})$ with multiplicity r_j, for each $j = 1, 2, \ldots, p$.

Continue...

- In [1], we see that there is no LPST between any pair of vertices in the corona product of two graphs, whenever the first graph has at least two vertices, as no vertex of the graph contains integer eigenvalues in its eigenvalue support.
- Ackelsberg et al. [1] showed that if H is a graph on $m \geq 1$ vertices, then $K_2 \circ H$ has LPGST between the vertices of K_2.

Continue...

- In [1], we see that there is no LPST between any pair of vertices in the corona product of two graphs, whenever the first graph has at least two vertices, as no vertex of the graph contains integer eigenvalues in its eigenvalue support.
- Ackelsberg et al. [1] showed that if H is a graph on $m \geq 1$ vertices, then $K_2 \circ H$ has LPGST between the vertices of K_2.
- Also they showed that, the corona product of the cocktail party graph $\overline{nK_2},\ n \geq 2$ with a single vertex graph admits LPGST.

Problem

To find the values of n for which the graph $K_n \circ \overline{K_m}$ exhibits Laplacian pretty good state transfer for all m.

References

[1] E. Ackelsberg, Z. Brehm, A. Chan, J. Mundinger, and C. Tamon. Laplacian state transfer in coronas. *Linear Algebra Appl.*, 506:154–167, 2016.

[2] L. Banchi, G. Coutinho, C. Godsil, and S. Severini. Pretty good state transfer in qubit chains—the heisenberg hamiltonian. *Journal of Mathematical Physics*, 58(3):032202, 2017.

[3] S. Barik, S. Pati, and B. Sarma. The spectrum of the corona of two graphs. *SIAM Journal on Discrete Mathematics*, 21(1):47–56, 2007.

[4] S. Bose, A. Casaccino, S. Mancini, and S. Severini. Communication in XYZ all-to-all quantum networks with a missing link. *Int. J. Quantum Inf.*, 7(4):713–723, 2009.

[5] G. Coutinho and H. Liu. No Laplacian perfect state transfer in trees. *SIAM J. Discrete Math.*, 29(4):2179–2188, 2015.

[6] G. de Morais Coutinho. *Quantum State Transfer in Graphs*. PhD thesis, University of Waterloo, 2014.

[7] E. Farhi and S. Gutmann. Quantum computation and decision trees. *Phys. Rev. A (3)*, 58(2):915–928, 1998.

THANK YOU !