Heterochromatic Geometric Transversals of Convex sets

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Definitions

- $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$
- $\mathcal{T}$ is a family of geometric objects in $\mathbb{R}^d$

Transversal

$\mathcal{T}$ is said to be a transversal of $\mathcal{F}$ if for every $F \in \mathcal{F}$, $\exists T \in \mathcal{T}$ such that

$$T \cap F \neq \emptyset.$$

- We say $\mathcal{T}$ pierces $\mathcal{F}$.

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(p, q)-property

$\mathcal{F}$ is said to satisfy (p, q)-property with respect to $\mathcal{T}$ if for every $\{F_1, F_2, \ldots, F_p\} \subset \mathcal{F}$, $\exists \{F_{i_j} \mid i_j \in [p], \forall j \in [q]\}$ and $\exists T \in \mathcal{T}$ such that

$$T \cap F_{i_j} \neq \emptyset, \ \forall j \in [q].$$

(p, q)-theorem

(p, q)-theorem[Alon-Kleitman, 1992]

For any three natural numbers $p \geq q \geq d+1$, $\exists c = c(p, q, d)$ such that if $\mathcal{F}$ is a collection of compact convex sets satisfying the (p, q)-property with respect to points in $\mathbb{R}^d$ then there exists a point transversal for $\mathcal{F}$ of size at most c.

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(p, q)-theorem for hyperplane transversal[Alon-Kalai, 1995]

For any three natural numbers $p \geq q \geq d + 1$, $\exists c' = c'(p, q, d)$ such that if $\mathcal{F}$ is a collection of compact convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property with respect to hyperplanes then there exists a hyperplane transversal for $\mathcal{F}$ of size at most c'.

Infinite (p, q)-theorem

- *k-flats:* k-dimensional affine space

$(\aleph_0, k+2)$-theorem [Keller-Perles, 2022]

Let $d \in \mathbb{N}, k \in \mathbb{Z}$ with $0 \leq k \leq d-1$. Suppose $\mathcal{F}$ is an infinite family of *nicely shaped* convex sets, satisfying $(\aleph_0, k+2)$-property with respect to k-flats, then the whole family $\mathcal{F}$ has a finite size k-transversal.

Colorful variant

- A sequence $\{B_n\}_{n\in\mathbb{N}}$ of convex sets in $\mathbb{R}^d$ is said to be *k-dependent* if there exists a k-flat K and $k+2$ distinct integers $i_1, \ldots, i_{k+2}$ such that for all $j \in [k+2]$, $K \cap B_{i_j} \neq \emptyset$.

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- If a sequence $\{B_n\}_{n\in\mathbb{N}}$ of convex sets is said to be *k-independent* if it is not k-dependent.

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- If a sequence $\{B_n\}_{n\in\mathbb{N}}$ of convex sets is said to be k-independent if it is not k-dependent.
- For each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a family of compact convex sets in $\mathbb{R}^d$.
- An infinite sequence $\{B_n\}_{n\in\mathbb{N}}$ is said to be a heterochromatic sequence of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ if there exists a strictly increasing sequence of natural numbers $\{i_n\}_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $B_n \in \mathcal{F}_{i_n}$.

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- If a sequence $\{B_n\}_{n\in\mathbb{N}}$ of convex sets is said to be k-independent if it is not k-dependent.
- For each $n\in\mathbb{N}$, $\mathcal{F}_n$ is a family of compact convex sets in $\mathbb{R}^d$.
- An infinite sequence $\{B_n\}_{n\in\mathbb{N}}$ is said to be a heterochromatic sequence of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ if there exists a strictly increasing sequence of natural numbers $\{i_n\}_{n\in\mathbb{N}}$ such that $\forall n\in\mathbb{N}$, $B_n\in\mathcal{F}_{i_n}$.
- $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is said to satisfy heterochromatic $(\aleph_0, k+2)$-property if every heterochromatic sequence from $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is k-dependent.

Our Work

- For each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a family of *nicely shaped* closed convex sets in $\mathbb{R}^d$.
- $k \in \mathbb{Z}$ with $0 \leq k \leq d - 1$

Heterochromatic $(\aleph_0, k+2)-$theorem [Chakraborty-Ghosh-N, 2023]

If $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ satisfies the heterochromatic $(\aleph_0, k+2)$-property, then all but finitely many $\mathcal{F}_n$'s can be pierced by finitely many k-flats.

Open Problem 1

- Can we establish the same result for general compact convex sets in $\mathbb{R}^d$, without any assumption of having nice shape?

Open Problem 2

Colorful (p, q)-theorem [Bárány-Fodor-Montejano-Oliveros-Pór, 2014]

For any $p, q, d \in \mathbb{N}$ with $p \geq q \geq d + 1$, $\exists M = M(p, q, d)$ such that the following holds: if $\mathcal{F}_1, \ldots, \mathcal{F}_p$ be finite families of convex sets in $\mathbb{R}^d$ satisfying colorful (p, q)-property with respect to points then there exists at least $q - d$ indices $i \in \{1, \ldots, p\}$ such that $\mathcal{F}_i$ has a point transversal of size at most M.

Open problem 2

Can we prove a Colorful (p, q)-theorem for hyperplane transversals?

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Can we prove a Colorful (p, q)-theorem for hyperplane transversals?

Thank You