Injective Coloring of Interval Graphs

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Overview

Injective Coloring

Graph Coloring

- A vertex k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$ and this vertex k-coloring is called a proper k-coloring if for every edge $uv \in E$, $f(u) \neq f(v)$.
- The chromatic number $\chi(G)$ of G is the minimum value of k for which G admits a proper k-coloring.

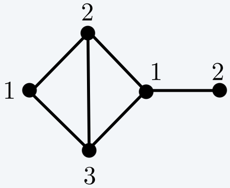

Figure: A Graph with Proper 3-Coloring

Injective Coloring

- An injective k-coloring of a graph G is a k-coloring of G such that no two vertices having a common neighbor receive the same color. In other words, for any two vertices $u, w \in N(v)$, $f(u) \neq f(w)$ for all $v \in V$.
- The injective chromatic number $\chi_i(G)$ of a graph G is the minimum value of k for which G admits an injective k-coloring.

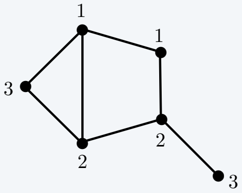

Figure: A Graph with Injective 3-Coloring

Known Results

- The concept of injective coloring was introduced by Hahn et al. in 2002.
- Hahn et al. [1] showed that $\Delta(G) \leq \chi_i(G) \leq \Delta(G)(\Delta(G)-1)+1$, where $\Delta(G)$ is the maximum degree of G and gave characterization achieving bound.
- Hell et al. [2] proved that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for chordal graphs by showing the NP-completeness for split graphs.
- They provided a polynomial time algorithm for the injective chromatic number of power chordal graphs.
- Panda et al. [3] showed that the injective chromatic number of proper interval graphs, threshold graphs, and $K_{1,3}$-free split graphs can be determined in linear time and the NP-completeness for the $K_{1,t}$-free split graphs, $t \geq 4$.

[1] G. Hahn, J. Kratochvil, J. Siran and D. Sotteau. On the injective chromatic number of graphs. *Discrete mathematics*, 256(1-2):179-192. 2002

[2] P. Hell, A. Raspaud, and J. Stacho. On injective coloring of chordal graphs. *In Latin American Symposium on Theoretical Informatics*, Springer, pages 520-530, 2008

[3] B. S. Panda, Priyamvada. Injective coloring of some subclasses of bipartite graphs and chordal graphs *Discrete Applied Mathematics*, 291:68-87, 2021.

Interval Graphs

Interval Graphs

- A graph G is an interval graph if it is the intersection graph of a family $\mathcal{F}$ of intervals in a linearly ordered set such as the real line.
- An interval ordering of G is an ordering of vertices $\sigma = (v_1, v_2, \ldots, v_n)$ of V with the property that if for $i \leq j \leq k$, $v_i v_k \in E$ then $v_j v_k \in E$.
- The following characterization of interval graphs is a key in many algorithms for interval graphs.

Theorem

[4] *A graph G is an interval graph if and only if G admits an interval ordering.*

[4] G. Ramalingam and C. P. Rangan. A unified approach to domination problems on interval graphs. *Information Processing Letters*, 27(5):271–274, 1988.

An Interval Graph

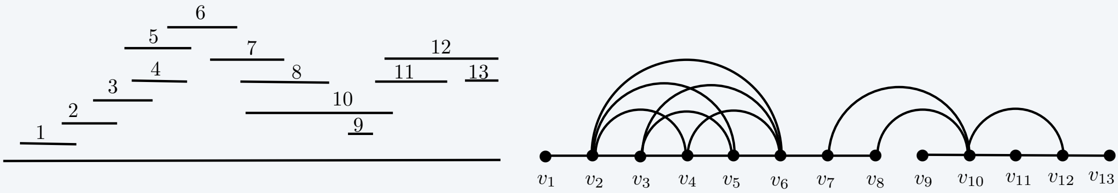

Figure: An Interval Graph with its interval representation and interval ordering

Our Contribution

Injective Coloring of Interval Graphs

Theorem

If G is an interval graph, then $\Delta(G) \leq \chi_i(G) \leq \Delta(G) + 1$.

Notation:

- $\sigma = (v_1, v_2, \ldots, v_n)$: an interval ordering.
- $S_{Max} = \{v_{r_1}, v_{r_2}, ..., v_{r_k}\}$: the set of all maximum degree vertices.
- $v_{l_i} = min(N[v_{r_i}])$ denote the minimum neighbor of v_{r_i} with respect to σ.
- $v_{f_i} = max(N[v_{r_i}])$ denote the maximum neighbor of v_{r_i} with respect to σ.
- $V_{Pendant} = \{v_{p_1}, v_{p_2}, \ldots, v_{p_a}\}$ be the set of all pendant vertices in G such that for all $j = 1, 2, \ldots, a$, $v_{p_j} \in N[v_{r_i}]$ for some $v_{r_i} \in S$.
- $V_{Lpendant} = \{v_{q_1}, v_{q_2}, \ldots, v_{q_b}\}$ be the set of all vertices which are not pendant in G but pendant in $G[N[v_{r_i}]]$ for some $v_{r_i} \in S$.

Lower Bound

$\chi_i(G) \geq \Delta(G)$

- $v_{r_i} \in S$ is a maximum degree vertex, each neighbor of v_{r_i} requires $\Delta(G)$ distinct colors.

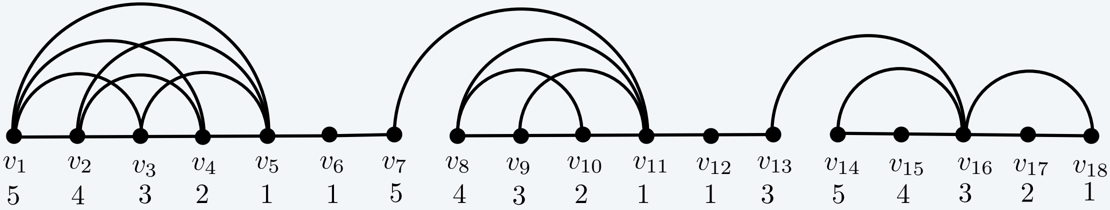

Figure: An Interval Graph with $\chi_i(G) = \Delta(G)$

Upper Bound

$\chi_i(G) \leq \Delta(G) + 1$

- Consider the ordering of the vertices $\sigma^{-1} = (v_n, \ldots, v_2, v_1)$.
- f is an injective coloring of G obtained by greedy injective coloring algorithm on σ^{-1}.

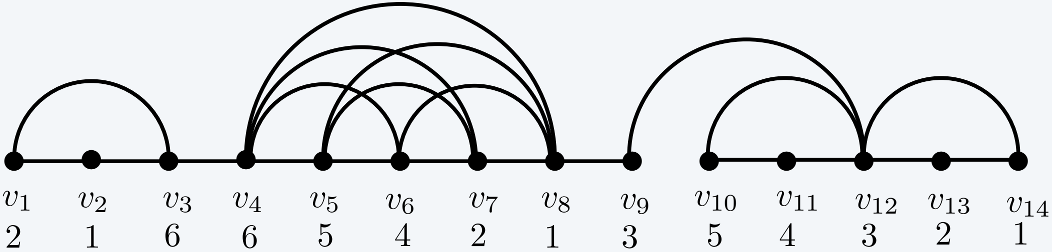

Figure: An Interval Graph with $\chi_i(G) = \Delta(G) + 1$

Theorem

- Consider a vertex v_i in G such that $f(v_i) = k$ where k is the maximum index.
- Let $v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}$, where $i_1 < i_2 < \ldots < i_{k-1}$ with respect to σ be the k vertices such that $\{f(v_{i_1}), f(v_{i_2}), \ldots, f(v_{i_{k-1}})\} = \{1, 2, \ldots, k-1\}$, which are forbidden for the vertex v_i.
- **Case 1.** $d(v_i) \geq k - 1$
  Now, $k \leq d(v_i) + 1 \leq \Delta(G) + 1$. Therefore, $\chi_i(G) \leq \Delta(G) + 1$ in this case.
- **Case 2.** $d(v_i) < k - 1$
  Now we have to find one vertex v_j such that $d(v_j) \geq k - 1$.
- Consider a vertex $v_j = max(N[v_{i_1}] \cap N[v_i])$ with respect to σ.

Illustration

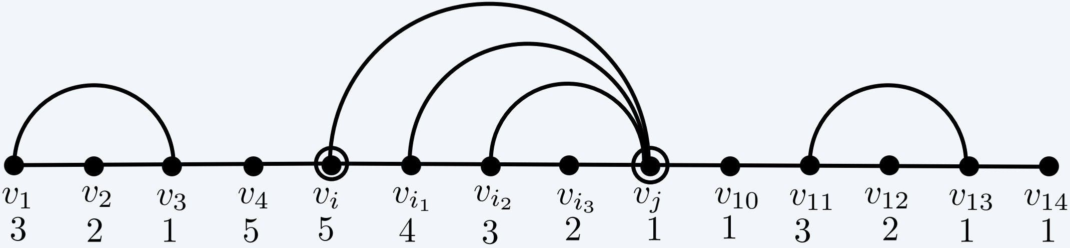

Figure: An Illustration of an Interval Graph with a vertex v_i with $d(v_i) < k - 1$ and v_j

Theorem

- **Subcase 1.** $i < i_1 < i_2 < \ldots < i_{k-1} \leq j$
- **Subcase 2.** $i < i_1 < i_2 < \ldots < i_r \leq j < i_{r+1} < i_{r+2} < \ldots < i_{k-1}$
- $i_s < j$
  We have, $i < i_s < j$ and $v_i v_s \in E$, by interval ordering $v_{i_s} v_j \in E$.
- $j < i_t$
  We have, v_{i_t} has a two length path with v_i and let v_i, v_t, v_{i_t} be a two length path between the vertices v_i and v_{i_t}. Then $t < j < i_t$ and $v_t v_{i_t} \in E$, by interval ordering, $v_j v_{i_t} \in E$.

Injective Coloring of Interval Graphs

- An L-vertex, R-vertex and LR-vertex is a vertex $v_{r_i} \in S$ that has exactly one neighbor v_{l_i} in $V_{Lpendent}$, one neighbor v_{f_i} in $V_{Lpendent}$, and two neighbors v_{l_i} and v_{f_i} in $V_{Lpendent}$ and no neighbor in $V_{Pendant}$ respectively.

Type-1 and Type-2 interval graph

G is said to be a Type-1 interval graph if it satisfies either of the following conditions:
C1: There exists a vertex $v_{r_i} \in S$ such that it has no pendant neighbors in $V_{Pendant}$ or $V_{Lpendsnt}$.
C2: There exist $(\alpha + 2)$-vertices, $\alpha \geq 0$ an R-vertex $v_{r_i} \in S$, α number of LR-vertices $v_{r_{j_1}}, v_{r_{j_2}}, \ldots, v_{r_{j_\alpha}} \in S$ and an L-vertex $v_{r_p} \in S$ such that $v_{f_i} = v_{l_{j_1}}, v_{f_{j_1}} = v_{l_{j_2}}, \ldots v_{f_{j_{\alpha-1}}} = v_{l_{j_\alpha}}$ and $v_{f_{j_\alpha}} = v_{l_p}$.
Otherwise, G is called a Type-2 interval graph.

Injective Coloring of Interval Graphs

This is the complete characterization of the interval graphs achieving the injective chromatic number $\Delta(G)$ and $\Delta(G) + 1$.

Theorem

If G is a TYPE-1 interval graph, then $\chi_i(G) = \Delta(G) + 1$.

Theorem

If G is a TYPE-2 interval graph, then $\chi_i(G) = \Delta(G)$.

Consequence

- Note that an optimal injective coloring of an interval graph can be obtained in $O(nm)$ time using the optimal injective coloring algorithm of the power chordal graph [5] as the class of interval graphs is a subclass of power chordal graphs.
- However, we proposed an $O(n+m)$ time algorithm to compute an optimal injective coloring of an interval graph.
- Further, we characterize the interval graphs for which $\chi_i(G) = \Delta(G)$ and $\chi_i(G) = \Delta(G) + 1$.

[5] G. Hahn, J. Kratochvil, J. Siran and D. Sotteau. On the injective chromatic number of graphs. *Discrete mathematics*, 256(1-2):179-192. 2002.

References

References

K. S. Booth, G. S. Lueker.
Testing for the consecutive ones property, interval graphs, and graph planarity using pq-tree algorithms.
Journal of computer and system Sciences, 13(3):335-379, (1976).

D. W. Cranston, S. J. Kim and G. Yu.
Injective colorings of sparse graphs.
Discrete Mathematics, 310(21): 2965-2973, 2010.

D. W. Cranston, S. J. Kim and G. Yu.
Injective colorings of graphs with low average degree.
Algorithmica, 60(3):553-568, 2011.

M. C. Golumbic.
Algorithmic graph theory and perfect graphs,
Elsevier, 2004.

References

G. Hahn, J. Kratochvil, J. Siran and D. Sotteau.
On the injective chromatic number of graphs.
Discrete mathematics, 256(1-2):179–192. 2002.

P. Hell, A. Raspaud, and J. Stacho,
On injective coloring of chordal graphs.
In Latin American Symposium on Theoretical Informatics, Springer, pages 520-530, 2008.

J. Jin, B. Xu, X. Zhang,
On the complexity of injective colorings and its generalizations,
Theoretical Computer Science, 491:119–126, 2013.

Borut Lužar, R. Škrekovski, M. Tancer,
Injective colorings of planar graphs with few colors,
Discrete Mathematics, 309(18):5636-5649, 2009.

References

B. S. Panda, Priyamvada
Injective coloring of some subclasses of bipartite graphs and chordal graphs.
Discrete Applied Mathematics, 291:68–87, 2021.

G. Ramalingam and C. P. Rangan.
A unified approach to domination problems on interval graphs.
Information Processing Letters, 27(5):271–274, 1988.

J. Song and J. Yue.
Injective coloring of some graph operations.
Applied Mathematics and Computation, 264:279–283, 2015.

Thank you for your attention

Note

- Injective coloring originated from the complexity theory of random access machines and it has application in the theory of error-correcting codes.
- A graph is an interval graph if and only if its maximal cliques can be ordered. such that each vertex that belongs to two of these cliques also belongs to all cliques between them in the ordering.
- We denote by G^k the k-th power of G, i.e., the graph obtained from G by making adjacent any two vertices in distance at most k in G. We call a graph G a power chordal graph if all powers of G are chordal.
- The injective chromatic number of a power chordal grph can be determined in polynomial ($O(mn)$) time.
- Coloring is also solvable for power chordal graphs in linear time.
- A graph G is power chordal if and only if any k-sun of G, $k \geq 4$, is suspended. Hence strongly chordal graphs are trivially power chordal graphs. Therefore interval graphs are also power chordal graphs.

Note

- If a graph G contains no n-sun, then Gk is a power chordal graph. Then G can be a tree, block graph, proper interval graph, interval graph, and strongly chordal graph.
  G is a power chordal graph, then G can contain a k-sun, but G^k do not contain a k-sun. A graph that contains a k-sun can be power chordal, but the k-sun itself is not.
- A graph $G = (V, E)$ of order n is an intersection graph if there exists a f bijection $f : V \to \mathcal{F}$, where $\mathcal{F}$ is a family of n sets such that $uv \in E$ if and only if $f(u) \cap f(v) \neq \emptyset$.
- **GREEDY INJECTIVE COLORING ALGORITHM**: Given an ordering $\alpha = (v_1, v_2, ..., v_n)$ of vertices of $G = (V, E)$, the greedy injective coloring algorithm assigns each vertex v_i the first available color that is not used by any vertex $v_j, j < i$ that has a common neighbor with v_i. The colors which are assigned to a vertex $vj, j < i$ that has a common neighbor with v_i are said to be forbidden for v_i.

Note

- **Subcase 1.** $i < i_1 < i_2 < \ldots < i_{k-1} \leq j$
  Since $v_i v_j \in E(G)$ and $i < i_1 < i_2 < \ldots < i_{k-2} < i_{k-1} \leq j$, by interval ordering $v_t v_j \in E(G)$ for all $t = i, i_1, i_2, \ldots, i_{k-2}$.
  Therefore, $d(v_j) \geq k - 1$. Hence, the claim is proved.
- **Subcase 2.** $i < i_1 < i_2 < \ldots < i_r \leq j < i_{r+1} < i_{r+2} < \ldots < i_{k-1}$ Since $v_i v_j \in E(G)$ and $i < i_1 < i_2 < \ldots < i_r \leq j$, by interval ordering $v_t v_j \in E(G)$ for all $t = i, i_1, i_2, \ldots, i_{r-1}$. Observe that each $v_{i_s} \in \{v_{i_{r+1}}, v_{i_{r+2}}, \ldots, v_{i_{k-1}}\}$ has a two length path with v_i and let v_i, v_s, v_{i_s} be a two length path between the vertices v_i and v_{i_s}. Note that, $j > s$ since $v_j = max(N[v_{i_1}])$. Since $v_s v_{i_s} \in E(G)$ and $s < j < i_s$, by interval ordering $v_j v_{i_s} \in E(G)$. Therefore, $v_j v_{i_s} \in E(G)$ for all $v_{i_s} \in \{v_{i_{r+1}}, v_{i_{r+2}}, \ldots, v_{i_{k-1}}\}$. Hence, $d(v_j) \geq r + (k - r - 1) = (k - 1)$. Therefore, $d(v_j) \geq (k - 1)$. Hence, the claim is proved.