Two interesting transit functions from I_G

Lekshmi Kamal K S

University of Kerala, India

YRF-CALDAM-2024

February 12, 2024

Transit function

Definition

A transit function on a nonempty finite set V is a function $R : V \times V \to 2^V$ satisfying the three transit axioms

$$(t1) \quad u \in R(u, v), \text{ for all } u, v \in V,$$

$$(t2) \quad R(u, v) = R(v, u), \text{ for all } u, v \in V,$$

$$(t3) \quad R(u, u) = \{u\}, \text{ for all } u \in V.$$

- If V is the vertex set of a graph G and R a transit function on V, then R is called a transit function on G.
- The underlying graph G_R of R is the graph (V, E), $uv \in E$ $(u \neq v)$ if and only if $R(u, v) = \{u, v\}$.

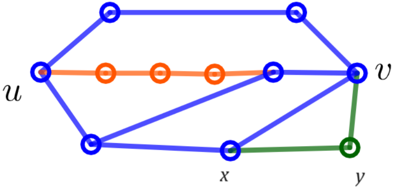
Problem

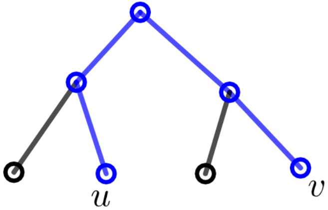
Characterize transit function R solely in terms of first order transit axioms on R. This problem becomes relevant as soon as the transit function is defined in terms of the structure σ, for instance, in the case of path functions.

Transit function on graph

- Interval function

$$I_G(u,v) = \{w \in V\text{: } w \text{ lies on some shortest } u,v \text{ - path in } G\} = \{w \in V : d(u,w) + d(w,v) = d(u,v)\}.$$





.

Characterization of interval function I

Theorem (H.M.Mulder, L.Nebeský, 2009)

Let $R : V \times V \longrightarrow 2^V$ be a function on V with the underlying graph G_R. Then $R = I_{G_R}$ if and only if R satisfying the axioms $(t1), (t2)$,

$(b2)$ $x \in R(u,v)$ *and* $y \in R(u,x) \implies y \in R(u,v)$,

$(b3)$ $x \in R(u,v)$ *and* $y \in R(u,x) \implies x \in R(y,v)$,

$(b4)$ $x \in R(u,v) \implies R(u,x) \cap R(x,v) = \{x\}$

$(s1)$ $R(u,\bar{u}) = \{u,\bar{u}\}, R(v,\bar{v}) = \{v,\bar{v}\}, u \in R(\bar{u},\bar{v})$ *and* $\bar{u},\bar{v} \in R(u,v)$, *then* $v \in R(\bar{u},\bar{v})$.

$(s2)$ $R(u,\bar{u}) = \{u,\bar{u}\}, R(v,\bar{v}) = \{v,\bar{v}\}, \bar{u} \in R(u,v)$, $v \notin R(\bar{u},\bar{v}), \bar{v} \notin R(u,v)$, *then* $\bar{u} \in R(u,\bar{v})$.

$(b1)$ $x \in R(u,v), x \neq v \implies v \notin R(u,x), \forall u,v \in V$.

Transit functions derived from I

- **Cycle Function** (C)

$$C(u,v) = \{u,v\} \cup \{w : w \in I(x,y) \text{ where } x,y \in I(u,v) \text{ and } I(x,y) \text{ is a cycle (which is clearly isometric)}\}.$$

$$C(u,v) = \{u,v\}, \text{ if } I(u,v) \text{ does not contain any cycle.}$$

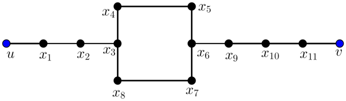
- **Stress Function** (S)
  - Let $u, v, x \in V$. The stress of the vertex x depending on a pair of vertex (u, v), denoted as $s_{uv}(x)$ and is defined as the number of distinct shortest paths from u to v which passes through x.
  - The stress function $S : V \times V \to 2^V$ is defined as

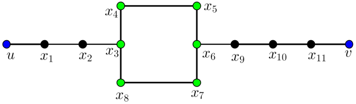
$$S(u,v) = \{x : x \in I(u,v) \text{ and } s_{uv}(x) \text{ is maximum }\}$$

$$= \{x : x \text{ lies in every shortest } u - v \text{ path }\}$$

.

Properties of C and S



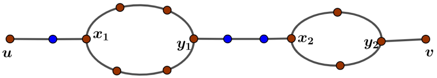


$$C(u,v) = \{u, v, x_3, x_4, x_5, x_6, x_7, x_8\}$$
$$S(u,v) = \{u, v, x_1, x_2, x_3, x_6, x_9, x_{10}, x_{11}\}$$

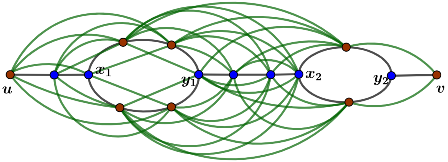
- $C(u,v) \subseteq I(u,v)$ & $S(u,v) \subseteq I(u,v)$
- $C(u,v) \cup S(u,v) = I(u,v)$
- Both S and C satisfies axioms $(b1)$, $(b2)$ and $(b4)$.
- $G_C \not\cong G$, $G_S \not\cong G$

$G_C \ncong G$, $G_S \ncong G$

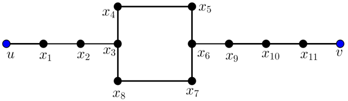
G :

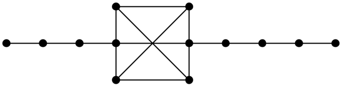


G_C:



- $C(u, v)$ contain only red colored vertices in both





Contrasting properties of C and S

Lemma

- $C(u, v) = \{u, v\}$ *if and only if* G *is a geodetic graph.*
- $C(u, v) = I(u, v)$ *if and only if* G *is a thick graph.*

Lemma

- $S(u, v) = \{u, v\}$ *if and only if* G *is a thick graph.*
- $S(u, v) = I(u, v)$ *if and only if* G *is a geodetic graph.*

Cycle Function Possess an FO characterization

(th): $x \in R(u,v), x \neq u,v \Rightarrow R(u,x) \cup R(x,v) \subseteq R(u,v), \forall u,v,x \in V$.

Theorem

Let R be a transit function on V. Then R satisfies axioms $(b3)$, $(s1)$, $(s2)$ and (th) if and only if $R = C_{G_R}$ and G_R is a thick graph.

Stress Function S

$(s3)$ $x \in R(u,v)$ if and only if $R(u,x) \cup R(x,v) = R(u,v)$

Lemma

- *Let G be a connected graph. Then the stress function of G satisfies the axioms $(s3)$.*
- *Let G be a graph with interval function I and stress function S. Then $x \in S(u,v)$ if and only if $I(u,x) \cup I(x,v) = I(u,v)$.*

PROBLEM:

- Is it possible to characterize S using a set of first-order axioms defined on R?
- Identify the properties satisfied by $\mathcal{C}$ but not S and vice-versa.

H.M. Mulder, L. Nebeský Axiomatic characterization of the interval function of a graph European J. Combin 30 2009 1172–1185

THANK YOU...