Cycles in 3-connected Planar Graphs

CHITRAK DAS

PRE-CONFERENCE

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Department of Computer Science(RKMVERI)

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Introduction

- Let H $\subset$ G; then G/H denotes the graph with V(G/H) = V(G-H) $\cup$ h and E(G/H)=E(G-H) $\cup$ [hy: y $\in$ V(G-H) and yy' $\in$ V(H)] We say that G/H is obtained from G by contracting H to the vertex h.
- A block of a graph G is a maximal 2-connected subgraph of G.
- A cycle is a facial cycle if it bounds a face of the graph.
- Open, Closed Disc
- Let H be a subgraph of a graph G: An H-bridge of G is a subgraph of G which either (1) is induced by an edge of E(G) - E(H) with both incident vertices in H or (2) is induced by the edges in a component of G from H and the edges of G from H to that component.For any H-bridge B of G; the attachments of B (on H) are the vertices in V(B) $\cap$ V(H).

Introduction-2

- A circuit graph is a pair (G,C), where G is a 2-connected plane graph and C is a facial cycle of G; such that, for any 2-cut S of G; every component of G-S contains a vertex of C.
- An annulus graph is a triple (G,C_1,C_2) where G is a 2-connected plane graph and C_1 and C_2 are facial cycles of G; such that, for any 2-cut S of G; every component of G - S contains a vertex of $C_1 \cup C_2$.

Introduction-Example

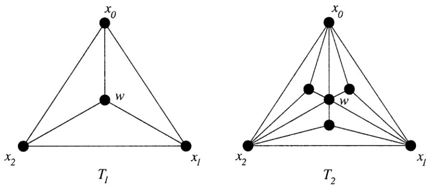

FIG. 1.

- First, we define a sequence of 3-connected plane graphs T_k as follows.Let T_1 be a plane graph isomorphic to K_4. Further, let V(T_1)=$[w, x_0, x_1, x_2]$ and let $x_0x_1x_2x_0$ be the outer cycle of T_1. Suppose that T_k is defined for some $k \geq 1$. Let T_{k+1} be the graph obtained from T_k as follows: In each inner face of T_k ; add a new vertex and join the new vertex to the vertices of T_k incident with that face. Graphs T_1 and T_2 are shown in Fig.1.

Introduction-Example

- The above construction, for any k≥1,T_k is a 3-connected plane graph with outer cycle $x_0x_1x_2x_0$ with circ(T_k) $< \frac{7}{2}n^{log_3 2}$
- Let α_k be the length of the longest x1-x2 path in T_k and β_k be the length of a longest x1-x2 path in $T_k - x_0$. By the construction of T_k, for i,j ∈[0,1,2] and i≠j, the length of a longest $x_i - x_j$ path in T_k is α_k and the length of a longest $x_i - x_j$ path in $T_k - ([x_0, x_1, x_2] - [x_i, x_j])$ is β_k.
- For i ∈ [0,1,2],let D^i denote the open disc in the plane bounded by the triangle in T_{k+1} induced by $[w, x_0, x_1, x_2] - [x_i]$. Let V^i denote the set of vertices in T_{k+1} contained in D^i,and let T^i be the plane subgraph of T_{k+1} induced by V^i ∪([w,x$_0, x_1, x_2] - [x_i]$).
- **Proposition :** $\alpha_k = 3 * 2^{k-1}$ and $\beta_k = 2^k$

Introduction-Important Theorems

- Let (G,C) be a circuit graph, and let x,y $\in$ V(C).We say that (G,xCy) is a strong circuit graph if, for any 2-cut S of G, S $\cap$ V(yCx-[x,y])$\neq \phi$
- Let R^+ denote the set of non-negative real numbers, and w:V(G) $\Longrightarrow$ R^+.For H$\subset$G,we write w(H) $= \Sigma_{v(H)} w(v)$.Define $w(\phi) = 0$.
- **Theorem**:- Let (G,xCy) be a strong circuit graph,and let w:V(G) $\Longrightarrow$ R^+. Then G contains an x-y path P such that $\Sigma_{v \in (P-y)} [w(v)]^{log_3 2} \geq [w(G-y)]^{log_3 2}$
- **Corollary**:- If (G,C) is a circuit graph and $e \in E(C)$.Then G contains a cycle T through e such that $|E(T)| \geq |v(G)|^{log_3 2}$
- **Corollary**:- Let G be a 3-connected planar graph,and let e$\in$E(G).Then G contains a cycle C through e such that $|E(C)| \geq |V(G)|^{log_3 2}$

Literature Review

- In 1963, Moon and Moser implicitly made the following conjecture by giving 3-connected planar graphs with $circ(G) \leq 9|V(G)|^{log_3 2}$ [1].
- In 2002, Chen and Yu proved that a 3-connected planar graph on n vertices contains a cycle of length at least $\Omega(n^{log_3 2})$ [2].
- In 1931 Whitney proved that every 4-connected triangulation of the plane contains a hamiltonian cycle.[3]
- In 1956 Tutte proved that every 4-connected planar graph contains a hamiltonian cycle.[4]

Problem Statement

- How many edge-disjoint cycles of length $\Omega(n^{log_3 2})$ can be guaranteed in a 3-connected planar graph.

References

- Simple paths on polyhedra.,J. W. Moon, L. Moser,1963[1]
- Long Cycles in 3-Connected Planar graph Guantao Chen, Xingxing Yu,2002[2]
- H. Whitney,A theorem on graphs,Ann of math[3]
- T. Tutte,A Theorem on planar graph[4]