# TWO HEURISTIC APPROACHES FOR SOME SPECIAL COLORINGS OF GRAPH

**Iztok Peterin**

FEECS, University of Maribor and IMFM Ljubljana

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Joint work with M. Anholcer, D. Božović, S. Cichacz, D. Gözüpek, B. Pawlik and D. Mesarič Štesl

# Literature


- M. Anholcer, S. Cichacz, I. Peterin, On b-acyclic chromatic number of a graph, Comput. Appl. Math. 42 (2023) #21 (20p).
- M. Anholcer, S. Cichacz, I. Peterin, On acyclic b-chromatic number of cubic graphs, in preparation.
- D. Božović, I. Peterin, D. Mesarič Štesl, On star b-chromatic number of a graph, in preparation.
- D. Gözüpek, I. Peterin, Grundy packing chromatic number of a graph, in preparation.
- B. Pawlik, I. Peterin, On Grundy acyclic chromatic number of a graph, in preparation.

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## Fact

Well known lower bound is $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the clique number of $G$.

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- We often analyze the worst case scenario to get some information about the quality of approximate or heuristic algorithm.
- The Grundy number $\Gamma(G)$ of a graph $G$ represents the worst case scenario for the greedy algorithm
- and the b-chromatic number $\chi_b(G)$ of $G$ in the case of recoloring algorithm.

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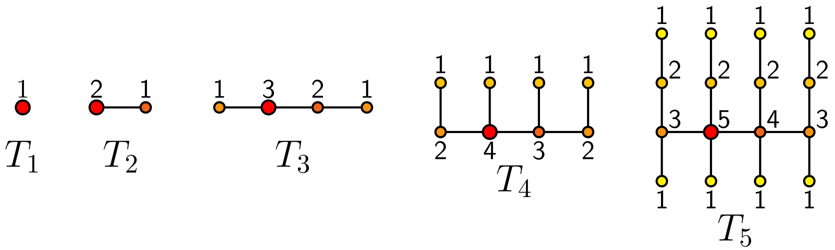
- One can find about 100 papers on this topic.

# An example on trees



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- For every vertex $v$ of color $k$ there exists a color $i_v \in \{1, \ldots, k-1\}$ such that

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**Theorem**

The **b-chromatic number** of a graph $G$, denoted $\chi_b(G)$, is the largest integer $k$ such that $G$ admits a proper $k$-coloring in which every color class contains at least one b-vertex.

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$$\chi(G) \leq \chi_b(G) \leq m(G).$$

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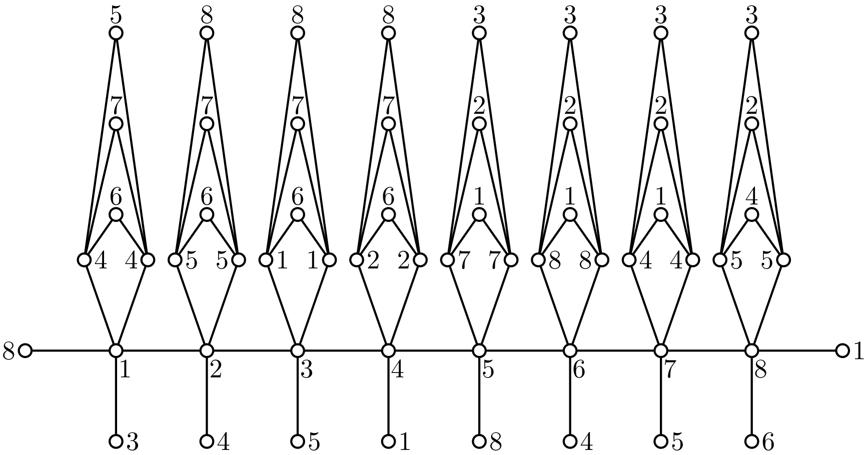
# Example




Slika: Graph $G_2$ with $\Delta(G_2) = 5 < 8 = A_b(G_2)$.

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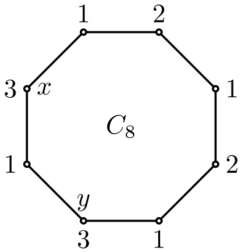


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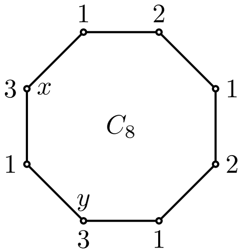


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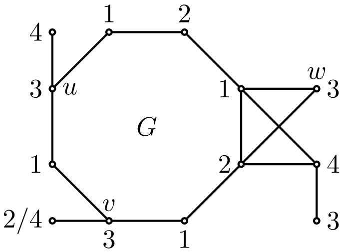


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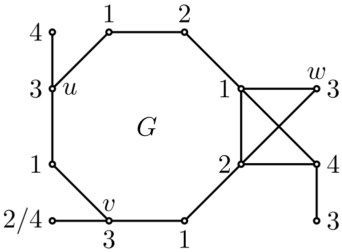


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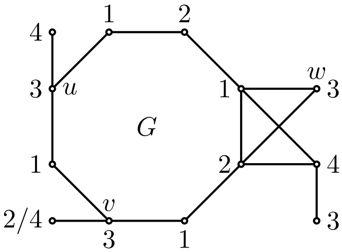


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- Hence, $C$ is of even length.
- Cycle $C$ is colored with exactly three colors say $k$, beside $i$ and $j$.
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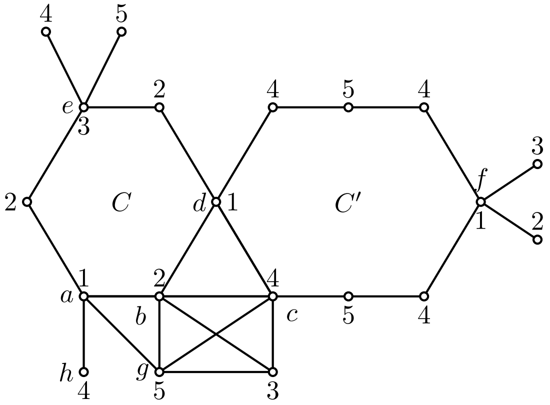
# Critical cycle systems




Slika: Cycles $C$ and $C'$ form a critical cycle system CCS$(1)$.

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## Corollary

The acyclic b-chromatic number $A_b(G)$ of a graph $G$ is the largest integer $k$, such that there exists an acyclic $k$-coloring, where every color class $V_i$, $i \in [k]$, contains an acyclic b-vertex.

# Some additional results

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For every positive integers $n, k, \ell$, where $k \geq 3$ and $\ell \geq 5$, we have

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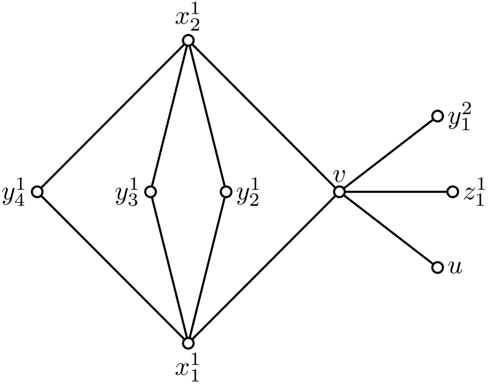
- This motivates the definition of the **acyclic degree** of $v$ as

$$d_G^a(v) = \max_{P \in \mathcal{P}(v)} \{(|A_0^P| + (|P| - 1) + \mathrm{elp}_G(v, P))\},$$

where $\mathcal{P}(v)$ is the family of all the weak partitions $P$ of $N_G(v)$ defined as above.

# An example




Slika: Graph $G$ with the optimal weak partition $A_0^P = \{u, z_1^1, y_1^2\}$,$A_1^P = \{x_1^1, x_2^1\}$, implying $|A_0^P| = 3$, $|P| - 1 = 1$, $\mathrm{elp}_G(v, P) = 3$ and $d_G^a(y_1^1) = 7$.

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For any graph $G$ we have $A_b(G) \leq m_a(G)$.

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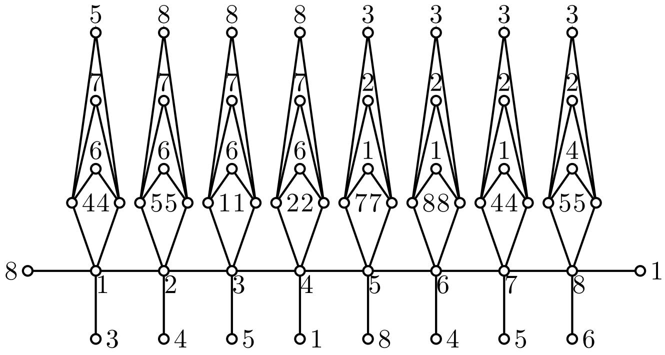

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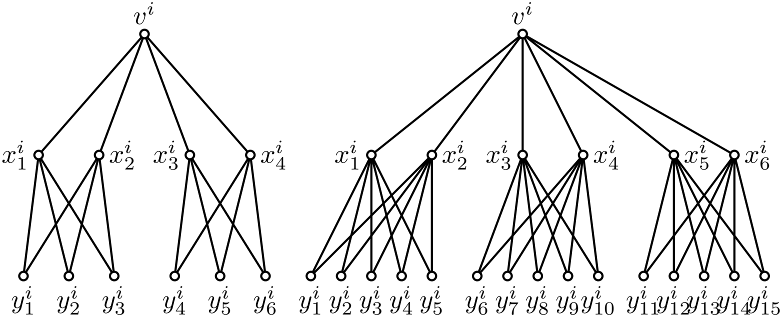
# A construction




Slika: Graphs $H_{2,i}$ and $H_{3,i}$.

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$$A_b(G \vee H) = \max\{A_b(G) + n_H, A_b(H) + n_G\}.$$

If $H \cong K_q$, then $A_b(G \vee H) = A_b(G) + q$.

**Corollary**

For every positive integers $k, \ell, m, n$, where $k, \ell \geq 5$, we have

- $A_b(K_{n,m}) = 1 + \max\{n, m\}$;

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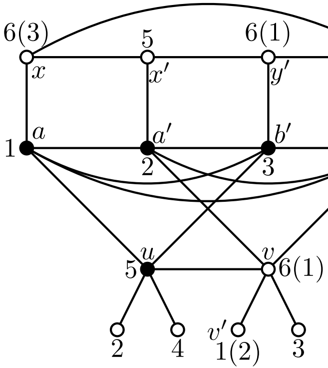
- $A_b(P_k \vee P_\ell) = A_b(P_k \vee C_\ell) = A_b(C_k \vee C_\ell) = 3 + \max\{k, \ell\}$.

# Relation between $\chi_b(G)$ and $A_b(G)$

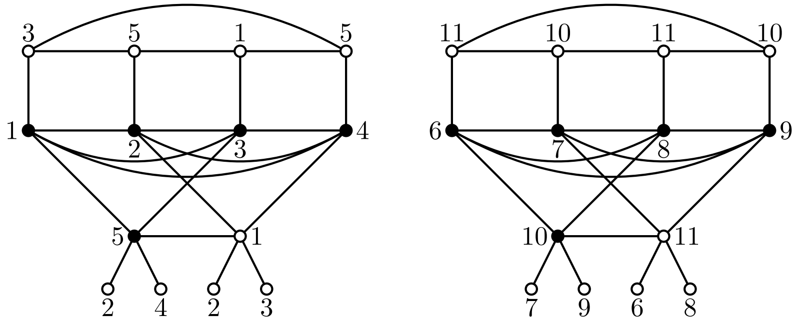
**Theorem**

There exists a graph $G$ where $A_b(G)$ is arbitrarily smaller that $\chi_b(G)$.



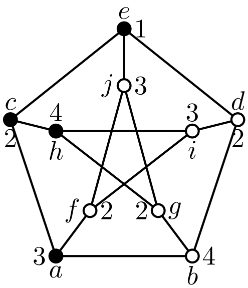

Slika: Graph $G$ for which $5 = A_b(G) < \chi_b(G) = 6$.

# Second example




Slika: Graph $G$ for which $10 = A_b(G) < \chi_b(G) = 12$.

# General result



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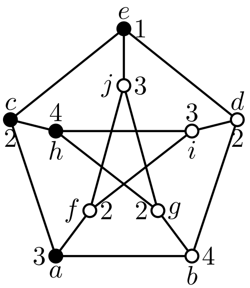


**Theorem**

For every cubic graph $G$ we have $A(G) \leq 4$.

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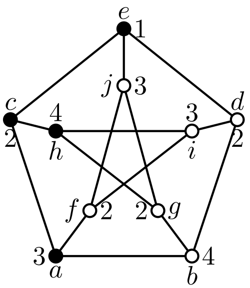


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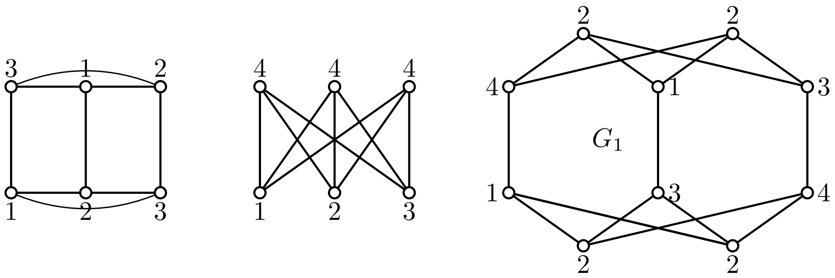
**Theorem**

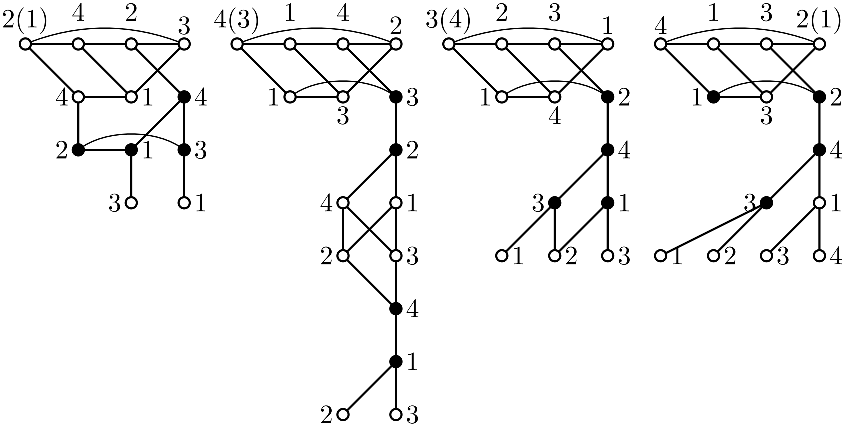
For every cubic graph $G$ but prism $K_2 \Box K_3$ we have $A_b(G) \geq 4$. Moreover, $A_b(K_2 \Box K_3) = 3$.

# Other exceptions from Jakovac and Klavžar



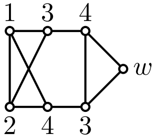

Slika: Graphs prism $K_2 \square K_3$, $K_{3,3}$ and $G_1$ and their acyclic b-colorings.

# Proof




Slika: Corrected b-colorings of Jakovac and Klavžar: first graph of the second line of Figure 14 and first, fourth and fifth graph from Figure 15; now acyclic.

# How many cubic graphs with $A_b(G) = 4$ exists?

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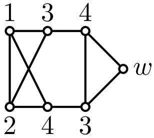


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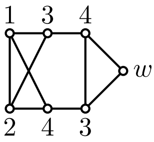


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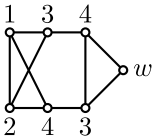


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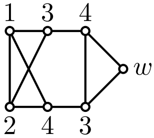


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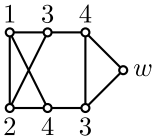

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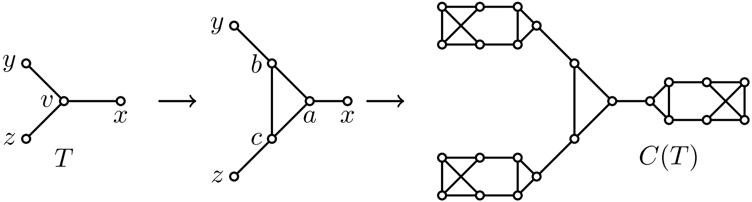
- Add a copy of $H_3$ for every leaf $\ell$ and amalgamate $\ell$ with $w$ from the copy of $H_3$ for $\ell$.

# A construction




Slika: Construction of graph $C(T)$ from a cubic tree $T \cong K_{1,3}$.

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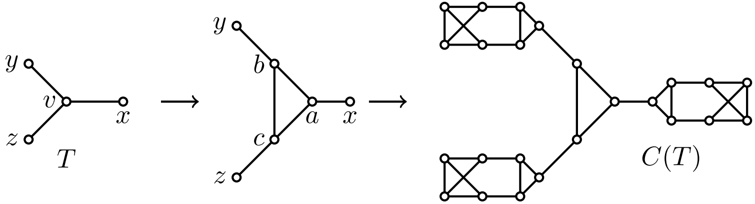


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**Theorem**

If $T$ is a cubic tree, then $A_b(C(T)) = 4$.

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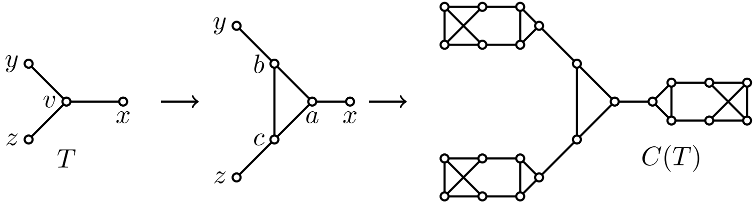


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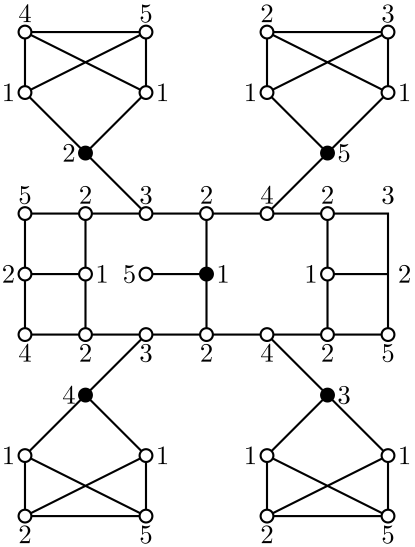
**Theorem**

If $T$ is a cubic tree, then $A_b(C(T)) = 4$.

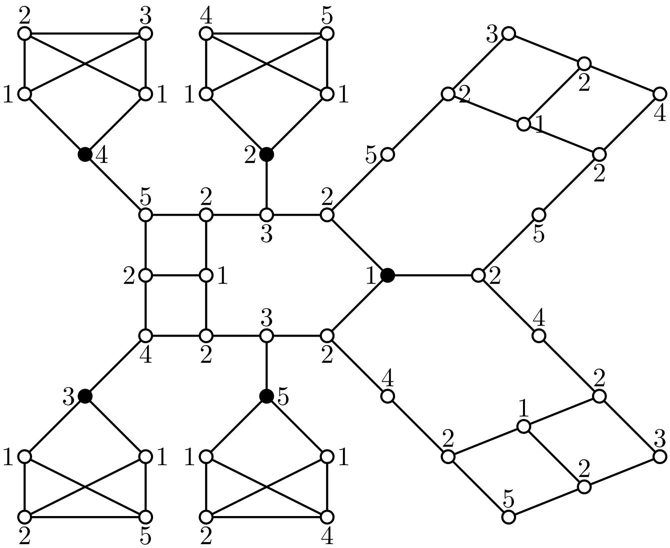
**Corollary**

The number of cubic graphs $G$ with $A_b(G) < m_a(G) = 5$ is not finite.

# Critical cycle system in cubic graph 1



# Critical cycle system in cubic graph 2



# Generalized Petersen graphs

- The generalized Petersen graphs $G(n,k)$, where $1 \leq k < n/2$, are the graphs on $2n$ vertices $\{x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}\}$

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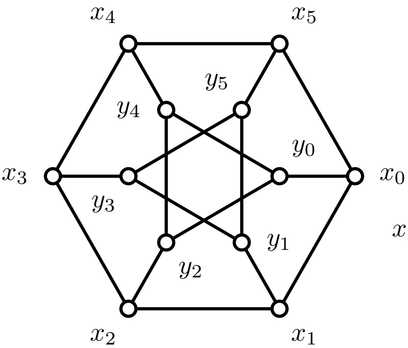


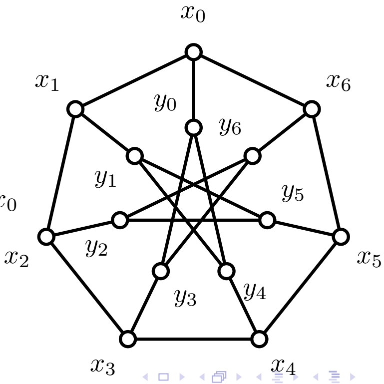

# Two results

**Theorem**

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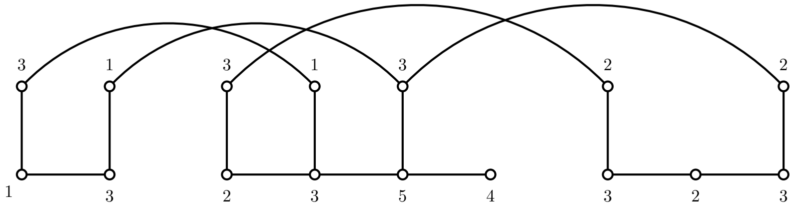
**Theorem**

$A_b(G(3,0)) = 4$ and $A_b(G(n,0)) = 4$ for $n \geq 4$.

# Proof idea for even $k$



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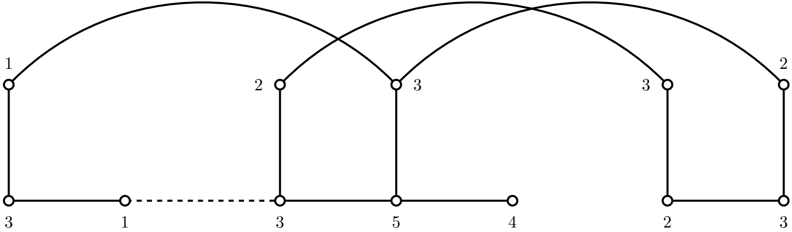

# Acycling coloring of $(0, j)$-prism

- The $(0, j)$-**prism** of order $2n$ for an even $j$ is the graph with two vertex disjoint cycles $R_n^i = v_0^i, \dots, v_{n-1}^i$ for $i \in \{1, 2\}$ of length $n$ called rims.

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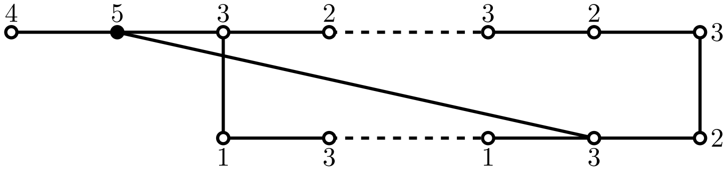
- $(0, j)$-prism is a cubic graph and is isomorphic to an $(0, -j)$-prism.
- Therefore we can assume that $0 \leq j \leq \frac{n}{2}$.

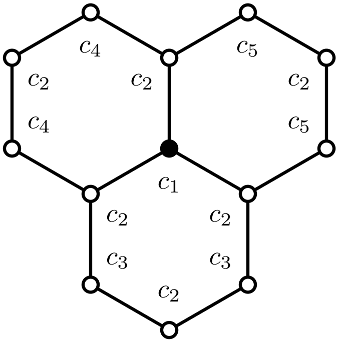
**Theorem**

If $j > 0$ and $n \geq 5(j + 2)$, then $A_b(Pr_n(0, j)) = 5$.

# Proof idea



# Honycomb lattice



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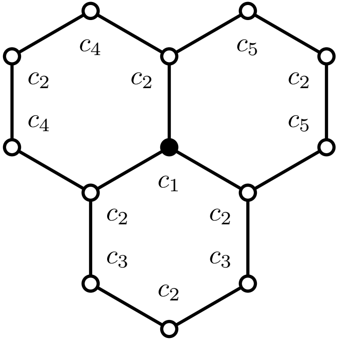


**Theorem**

Let $G$ be a benzenoid graph. Assume there are five internal vertices $v_j$, $j \in [5]$ in $G$, such that for each $1 \leq i < j \leq 5$ we have $d(v_i, v_j) \geq 4$. If every spanning tree of the distance graph $D_G(\{v_1, v_2, v_3, v_4, v_5\})$ contains at least one edge weighted with 5, then $A_b(G) = 5$.

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- **Acyclic Grundy chromatic number** $\Gamma_a(G)$ of $G$ is the maximum number of colors obtained by the mentioned procedure.

# Corona of graphs

## Proposition

For graphs $G$ and $H$ we have

$$\Gamma_a(G \odot H) = \Gamma_a(G) + \Gamma_a(H)$$

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## Theorem

For every natural number $k$ there exists a graph $G$ such that

$$\Gamma_a(G) - \Gamma(G) = k.$$

# Join of graphs

**Theorem**

For graphs $G$ and $H$ we have

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**Corollary**

For every positive integers $m, n$ we have

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# Join of graphs

## Theorem

For graphs $G$ and $H$ we have

$$\Gamma_a(G \vee H) = \max\{\Gamma_a(G) + |V(H)|,\ \Gamma_a(H) + |V(G)|\}.$$

## Corollary

For every positive integers $m, n$ we have

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- $\Gamma_a(K_m \vee \overline{K_n}) = m + 1$.

# Upper bound

**Theorem**

For every positive integer $\Delta$ there exists a graph $G$ such that $\Delta = \Delta(G)$ and

$$\Gamma_a(G) \leqslant \begin{cases} \frac{3\Delta^2+13}{8} & \text{if } \Delta \text{ is odd,} \\ \frac{3\Delta^2+2\Delta+8}{8} & \text{if } \Delta \text{ is even.} \end{cases}$$

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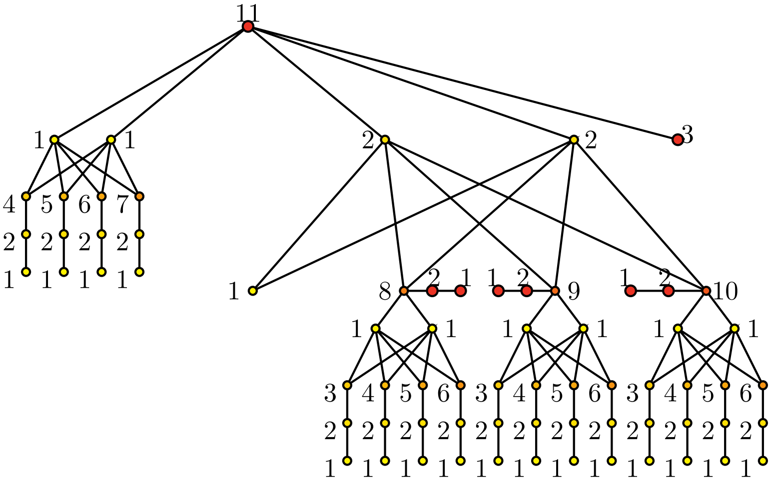
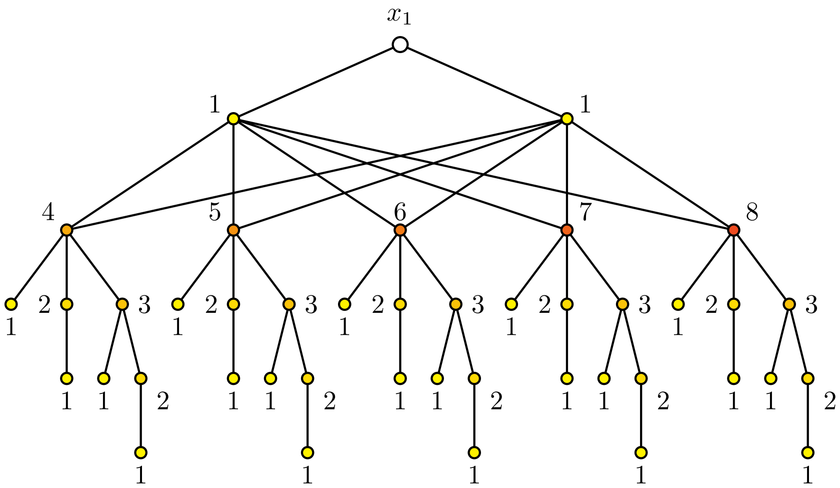
## Table

| $\Delta(G)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Gamma_a(G)$ | 2 | 3 | 5 | 8 | 11 | 16 | 20 | 27 | 32 | 41 | 47 | 58 | 65 |

# Odd maximum degree



# Even maximum degree



# $\Gamma(G)$ can be bigger than $\Gamma_a(G)$

**Theorem**

There exists an infinite family of graphs $G_1$, $G_2$, ... such that

$$\Big(\Gamma(G_k) - \Gamma_a(G_k)\Big) \to \infty$$

as $k \to \infty$.

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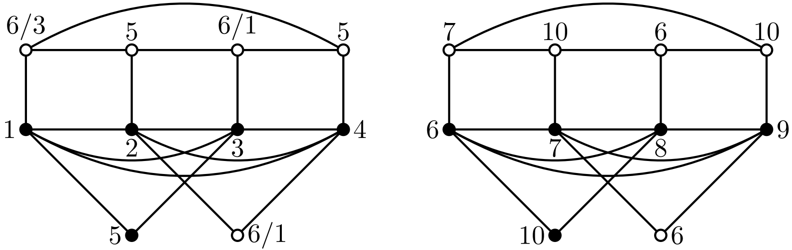


Slika: Graph $G$ such that $\Gamma(G) \geq \Gamma_a(G)$.

# Packing chromatic number

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- So, an $i$-th color class of a packing coloring represents $i$-packing of $G$.
- The **packing chromatic number** $\chi_p(G)$ is the minimum integer $k$ for which there exists a packing $k$-coloring of $G$.
- We adopt a greedy algorithm to produce a packing chromatic number of $G$.

# Heuristic algorithm for packing coloring

## Algorithm

- **Input:** Graph $G$ and every vertex with $|V(G)|$ dimensional array with 1s for every vertex.

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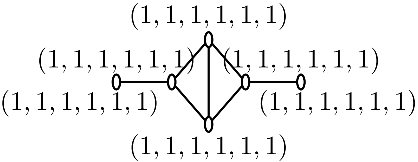
## Theorem

Algorithem computes a packing coloring of a given graph $G$ in $\mathcal{O}(mn^2)$ time, where $n = |V(G)|$ and $m = |E(G)|$.

# An example



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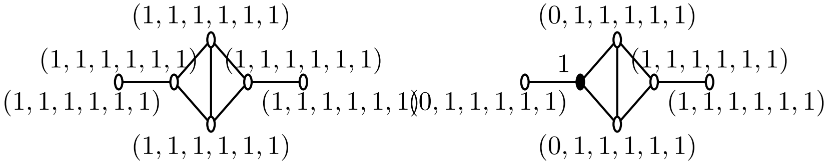
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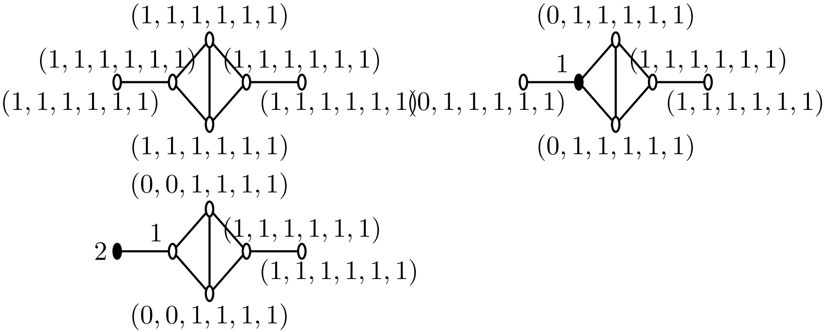

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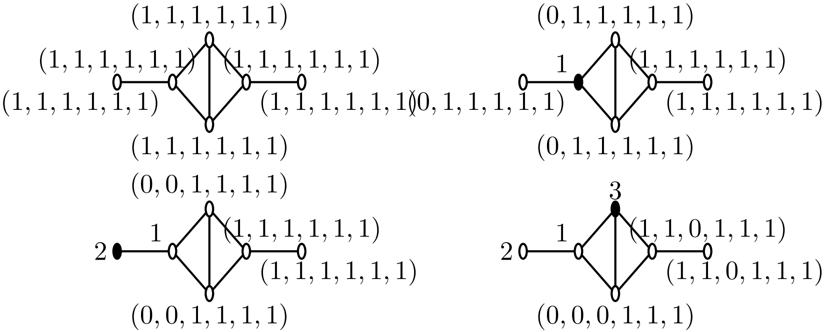
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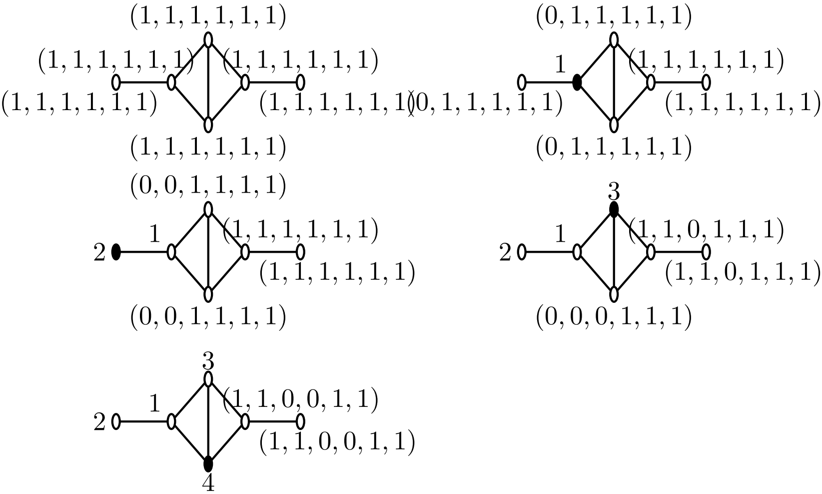

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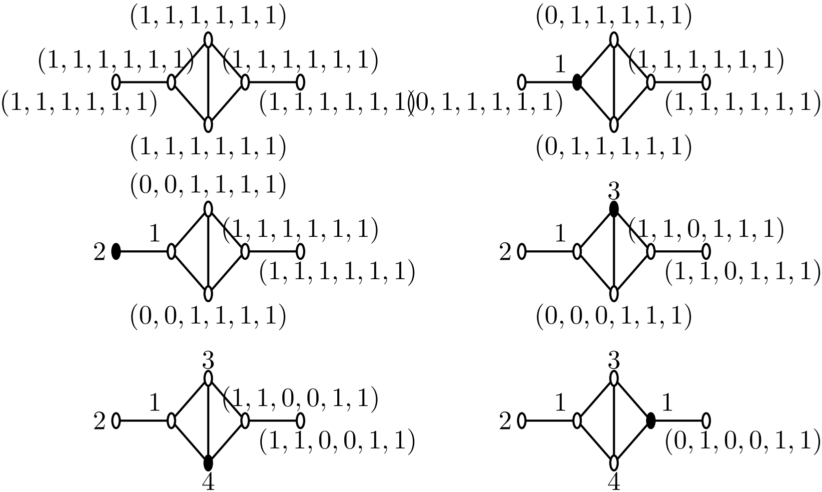
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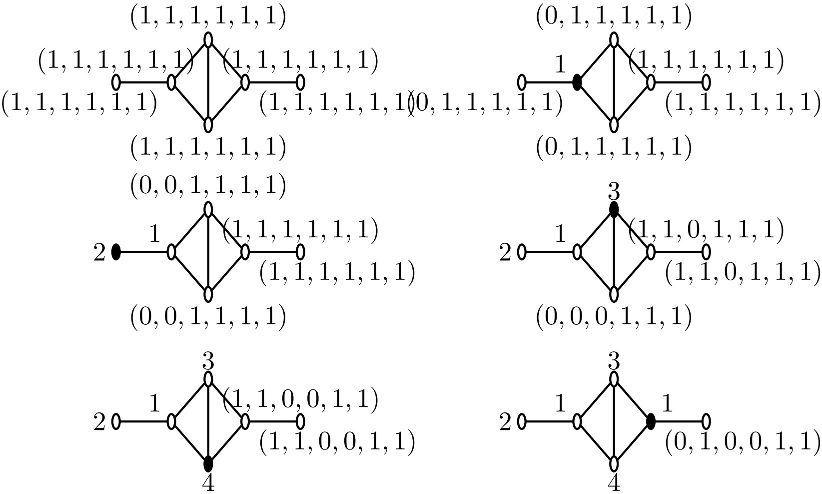
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- Alternative description of $\Gamma_p(G)$ is just the maximum number of colors in a packing coloring, such that every vertex of color $i \geq 2$ has a vertex of color $j$ at distance at most $j$ for every $j \in \{1, \ldots, i-1\}$.

# Polynomial transformation

- For a graph $G$ we denote by $G^\ell$ a graph with $V(G^\ell) = \{v^\ell : v \in V(G)\}$ and $E(G^\ell) = \{u^\ell v^\ell : d_G(u,v) \leq \ell\}$.

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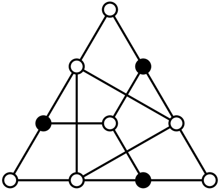
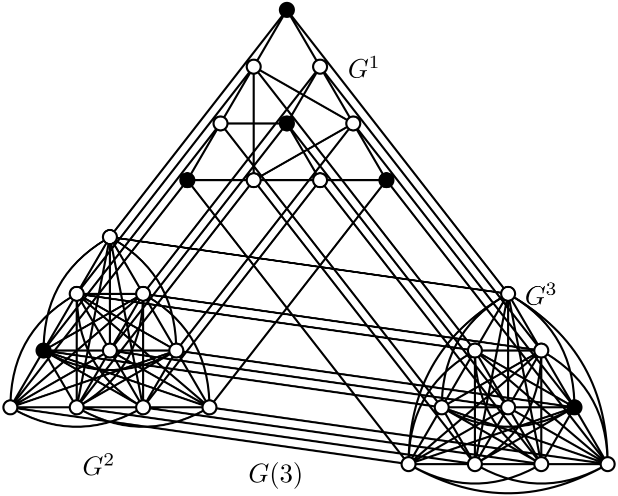
- $v^1, \ldots, v^k$ induces a clique $Q_v$ in $G(k)$ and that every independent set of $G(k)$ contains at most one of them.

# An example



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**Lemma (Argiroffo et al.)**

Let $G$ be a graph on $n$ vertices and $k \in \{1, \ldots, n\}$. A graph $G$ admits a packing $k$-coloring if and only if there exists an independent set of cardinality $n$ in $G(k)$.

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- However, the condition of last lemma is not always fulfilled.
- Therefore we introduce a **dense maximization procedure** or DMP for short of an independent set $A$ of $G(k)$ of cardinality $n$.

# Dense maximization procedure

- If $A^i = A \cap G^i$ is a maximal independent set of $G^i - \cup_{j=1}^{i-1} A^j$ for every $i \in \{1, \ldots, k-1\}$, then we are done.

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## Theorem

Let $G$ be a graph with $n = |V(G)|$. If $\mathcal{I}$ is a set of all independent sets of $G(k)$ of cardinality $n$ for any possible integer $k \leq n$, then

$$\Gamma_p(G) = \max_{A \in \mathcal{I}} \{DMP(A)\}.$$

# Computational complexity of $\Gamma_p(G)$

**Corollary**

If $G$ is a graph, then $\Gamma_p(G) \leq n - i(G) + 1$ wher $i(G)$ denotes the lower independence number.

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## PGC problem

PACKING GRUNDY COLORING PROBLEM
INSTANCE: A graph $G$ on $n$ vertices and an integer $1 \leq k \leq n$.
QUESTION: Is $\Gamma_p(G) \leq k$?

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**Theorem**

PGC problem is NP-complete even when $G$ is restricted to bipartite graphs, to line graphs, to circle graphs, to unit disk graphs, or to planar cubic graphs.

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**Lemma**

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**Theorem**

A connected graph $G$ on $n$ vertices has $\Gamma_p(G) = n - 1$ if and only if all the following statements hold

(i) $i(G) = 2$.

(ii) $\mathrm{diam}(G) \leq 4$.

(iii) If $\mathrm{rad}(G) = 3$, then there exist an $i(G)$-set $\{x, y\}$ with $d(x, y) = 3$ and $w \in N(x)$ such that $d(w, z) \leq 2$ for every vertex $z \in N(y)$.

(iv) If $\mathrm{rad}(G) = 2$ and $\mathrm{diam}(G) = 4$, then there exists an $i(G)$-set that avoids one central vertex and one additional non-diametrical vertex.

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$\Gamma_p(G \vee H) = |V(G)| + |V(H)| - \min\{i(G), i(H)\} + 1$. In particular, for $s, t \geq 1$, $p, r \geq 4$ and $n \geq 2$,

- $\Gamma_p(K_{s,t}) = s + t - \min\{s, t\} + 1$;

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# Graphs with $\operatorname{diam}(G) = 2$

## Theorem

Let $G$ be a graph and let $\mathcal{I}$ be a family of all maximal independent sets of $G$. If $\operatorname{diam}(G) = 3$, then $\Gamma_p(G) = |V(G)| - m(G) + 2$ where

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**Relation $Q_s$**

We say that coloring $c'$ is in relation $Q_s$ with coloring $c$, that is $c' Q_s c$.

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- Hence $S_b(G)$ is a kind of a dual of $S(G)$.

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Color $\ell \neq c(v)$ is blocked for vertex $v \in V(G)$ if

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A vertex $v \in V(G)$ is a **star b-vertex** if every color $\ell \in [k]$ is blocked.

# Characterization

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A star $k$-coloring $c$ is a minimal element of $\prec_s$ if and only if every color class $V_i$, $i \in [k]$, contains a b-star vertex.

**Corollary**

The star b-chromatic number $S_b(G)$ of a graph $G$ is the largest integer $k$, such that there exists a star b-coloring with $k$ colors, where every color class $V_i$, $i \in [k]$, contains a b-star vertex.

# Star degree

## b-vertex

- $N_2(v) = \{u \in V(G) : d(v,u) = 2\}$ and $N_3(v) = \{u \in V(G) : d(v,u) = 3\}$.

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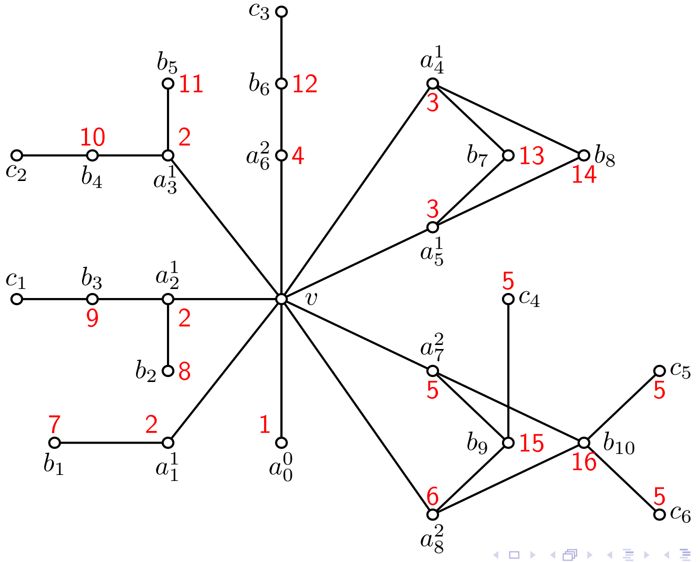
## Theorem

Maximum $d_G^s(v)$ of colors can be blocked for $v \in V(G)$.

## An example


$c_3$
$b_5$ 11
$b_6$ 12
$a_4^1$ 3
10 2
$c_2$ $b_4$ $a_3^1$
$a_6^2$ 4
$b_7$ 13
$b_8$ 14
3 $a_5^1$
$c_1$ $b_3$ $a_2^1$
9 2
$v$
5 $c_4$
$b_2$ 8
$a_7^2$ 5
$c_5$ 5
7 2 1
$b_1$ $a_1^1$ $a_0^0$
$b_9$ 15
$b_{10}$ 16
6 $a_8^2$
5 $c_6$

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## Star m-degree

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## Theorem

For any graph $G$ we have $S_b(G) \leq m_s(G)$.

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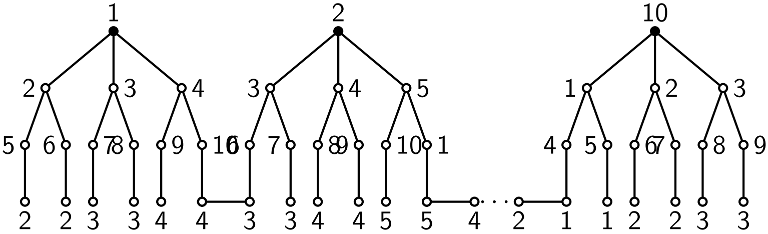


Slika: Graph $G_3$ and its 10-star b-coloring.

# Paths and cycles

## Proposition

Let $P_n$ be a path on $n$ vertices. Then

$$S_b(P_n) = \begin{cases} 1 & ; n = 1 \\ 2 & ; 2 \leq n \leq 3 \\ 3 & ; 4 \leq n \leq 7 \\ 4 & ; 8 \leq n \leq 22 \\ 5 & ; n \geq 23. \end{cases}$$

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## Proposition

Let $C_n$ be a cycle on $n \geq 3$ vertices. Then

$$S_b(C_n) = \begin{cases} 3 & ; n \leq 9 \\ 4 & ; 10 \leq n \leq 19 \\ 5 & ; n \geq 20. \end{cases}$$

# Relation between $m_s(G)$ and $S_b(G)$

**Theorem**

There exists an infinite family of graphs $G_1, G_2, \ldots$ such that $(m_s(G_n) - S_b(G_n)) \to \infty$ as $n \to \infty$.

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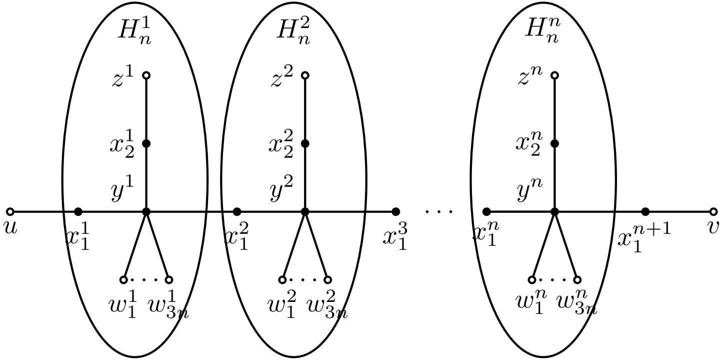


Slika: The infinite family of graphs $G_n$.

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- $S_b(K_{n,m}) = 1 + \max\{n, m\}$;

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- $S_b(F_k) = S_b(P_{k-1}) + 1$;
- $S_b(K_n \vee \overline{K}_m) = n + 1$.

# Two more relations

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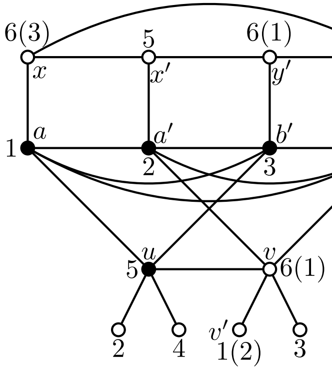
## Corollary

There exists an infinite family of graphs $G_1, G_2, \ldots$ such that $(S_b(G_n) - \chi_b(G_n)) \to \infty$ as $n \to \infty$.

# Relation between $\chi_b(G)$ and $A_b(G)$

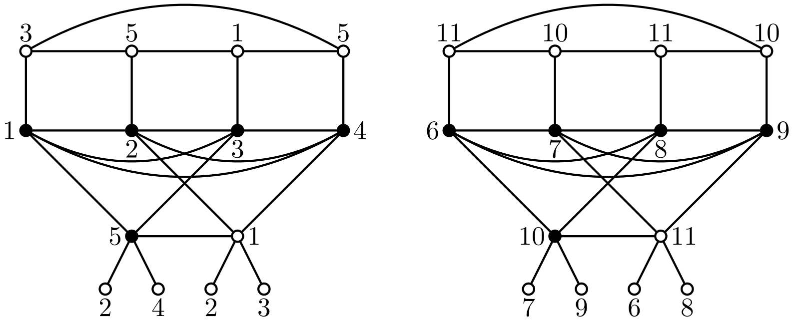
## Theorem

There exists a graph $G$ where $S_b(G)$ is arbitrarily smaller that $\chi_b(G)$.




Slika: Graph $G$ for which $5 = A_b(G) < \chi_b(G) = 6$.

# Second example




Slika: Graph $G$ for which $10 = S_b(G) < \chi_b(G) = 12$.

# End

THANK YOU FOR YOUR ATTENTION!