Total Coloring of Graphs

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Overview

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2. Edge Coloring of Graphs
3. Total Coloring Problems for some class of graphs
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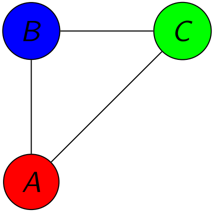
Definitions and Background

Vertex Coloring

Vertex Coloring

A proper vertex coloring of a graph G is an assignment of colors to the vertices such that no two adjacent vertices receive the same color.

The Chromatic Number of G, denoted by $\chi(G)$, is the minimum number of colors required for a proper vertex coloring of a graph G.



$$\chi(K_3) = 3$$

Vertex Coloring

Theorem (Brooks 1941)

For a graph G with maximum degree $\Delta(G)$,

$$\chi(G) \leq \begin{cases} \Delta(G)+1 & \textit{if } G \textit{ is a complete graph or an odd cycle,} \\ \Delta(G) & \textit{otherwise.} \end{cases}$$

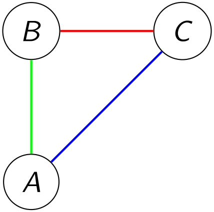
Edge Coloring of Graphs

Edge Coloring

Edge Coloring

An proper Edge Coloring of a graph G is an assignment of colors to its edges such that no two edges incident on a vertex receive the same color.

The Edge Chromatic Number of G, denoted by $\chi'(G)$, is the minimum number of colors required for a proper edge coloring of the graph G.



$$\chi'(K_3) = 3$$

Edge Coloring

Theorem (Vizing 1964)

For a graph G with maximum degree $\Delta(G)$,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

Definition

G is of Class 1 if $\chi'(G) = \Delta$ and is of Class 2 if $\chi'(G) = \Delta + 1$

Sufficient condition for Class I

Theorem

(Vizing) If G_Δ , (where G_Δ , core of G, is the subgraph induced by all major vertices of G), is forest, then G is of Class 1.

Edge Coloring Classification

Definition

For a vertex $v \in V(G)$, we denote the degree of v by $d_G(v)$, the number of vertices of maximum degree joined with v by $d_G^*(v)$, and $d^*(G) = \min_{v \in V(G)} d_G^*(v)$.

Theorem (Niessen and Volkman[1990])

If G is a graph of odd order $2n+1$ such that

1. $E(G) \leq n\Delta(G)$*, that is, G is not overfull, and*
2. $\delta(G) \geq n + |V(G_\Delta)| + d^*(G)$.

then $\chi'(G) = \Delta(G)$.

Theorem (Niessen and Volkman[1990])

If G is a graph of even order $2n$ such that $\delta(G) \geq n + |V(G_\Delta)| - 2$, then $\chi'(G) = \Delta(G)$.

Some Conjectures

Definition

A graph is called a 1-factorizable is it can be factored into perfect matchings.

Conjecture (Hilton and Chetwynd[1985])

For a regular simple graph G on $2n$ vertices of degree $d(G)$, if $d(G) \geq n$ then G is 1-factorizable.

Theorem (Niessen and Volkman[1990])

If G is a regular graph with $d(G) \geq \frac{1}{2}(\sqrt{7}-1)|V(G)|$, then G is 1-factorizable. ($d(G) \geq 1.64n$)

Total Coloring

Total Coloring

A total coloring of a graph G is a coloring of the vertices and the edges such that:

- no two adjacent vertices receive the same color (vertex coloring),
- no two incident edges receive the same color (edge coloring),
- **no edge receives the same color as one of its endpoints.**

Total Coloring

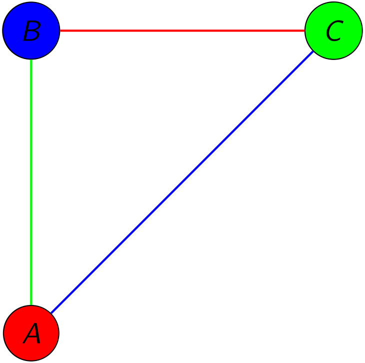
Total Coloring

A total coloring of a graph G is a coloring of the vertices and the edges such that:

- no two adjacent vertices receive the same color (vertex coloring),
- no two incident edges receive the same color (edge coloring),
- **no edge receives the same color as one of its endpoints.**

The Total Chromatic Number of G, denoted by $\chi_T(G)$ or $\chi''(G)$, is the minimum number of colors required for a total coloring a graph G.

Total Coloring



$$\chi''(K_3) = 3$$

Total Coloring

Total Coloring Conjecture (Behzad 1965, Vizing 65)

Let $\Delta(G)$ is the maximum degree of the graph G.

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$$

Total Coloring Classification Problem

Total Coloring Conjecture (Behzad 1965, Vizing 65)

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$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$$

Total Coloring Classification Problem

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Let $\Delta(G)$ is the maximum degree of the graph G.

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For a graph G which satisfies the Total Coloring Conjecture, there is a natural classification into two categories:

Total Coloring Classification Problem

Total Coloring Conjecture (Behzad 1965, Vizing 65)

Let $\Delta(G)$ is the maximum degree of the graph G.

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$$

For a graph G which satisfies the Total Coloring Conjecture, there is a natural classification into two categories:

$$G \text{ is of } \begin{cases} \text{Type 1} & \text{if } \chi'' = \Delta(G) + 1, \\ \text{Type 2} & \text{if } \chi'' = \Delta(G) + 2. \end{cases}$$

The class of Regular graphs is complete for TCC

- If H is a subgraph of G, then $\chi''(H) \leq \chi''(G)$.
- (Konig's Theorem) Every graph G with maximum degree k can be embedded into a k-regular graph.
- So, if TCC holds for regular graphs, then it holds for every graph. So, the class of regular graphs is complete for TCC.

Total Graph of a Graph

Definition

Total Graph of $G = (V, E)$ is $G_T = (V_T, E_T)$, $V_T = V \cup E$ and $E_T = \{ve | v \text{ is incident on } e\} \cup \{uv | uv \in E\} \cup \{ef | e \text{ and } f \text{ are adjacent in } G\}$.

Note: $\chi''(G) = \chi(G_T)$.

Theorem

The Indpendence Number of Total Graph satisfies:
$\alpha(G_T) = \alpha_T(G) \leq \alpha(G) + \lfloor \frac{|V(G)| - \alpha(G)}{2} \rfloor$

Comparison for Complete Graph

For a complete graph K_n with maximum degree $\Delta(K_n) = n - 1$,

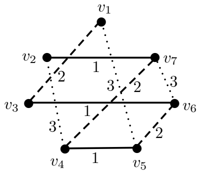
$$\chi(K_n) = \Delta(K_n) + 1,$$

$$\chi'(K_n) = \begin{cases} \Delta(K_n) & \text{if } n \text{ is even,} \\ \Delta(K_n) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

$$\chi''(K_n) = \begin{cases} \Delta(K_n) + 2 & \text{if } n \text{ is even,} \\ \Delta(K_n) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Totoal Coloring of K_n, n odd

- For odd n, arrange the vertices as the vertices of a regular n-gon. The set $\{v_1, v_2v_n, v_3v_{n-1}, v_4v_{n-2}, \ldots,\}$ is one color class, The set $\{v_2, v_3v_1, v_4v_n, v_5v_{n-1}, \ldots,\}$ is another color class and so on.



- So $\chi''(K_n) = \Delta(K_n) + 1$, if n is odd. So, K_n, n odd, is Type-I graph.

Total Coloring of K_n, n is even

- For even n, $\chi''(K_n) \leq \chi''(K_{n+1}) = n + 1 = \Delta(K_n) + 2$.

Total Coloring of K_n, n is even

- For even n, $\chi''(K_n) \leq \chi''(K_{n+1}) = n + 1 = \Delta(K_n) + 2$.
- How to get lower bound?

Total Coloring of K_n, n is even

- For even n, $\chi''(K_n) \leq \chi''(K_{n+1}) = n + 1 = \Delta(K_n) + 2$.
- How to get lower bound?
- Counting Technique

Total Coloring of K_n, n is even

- For even n, $\chi''(K_n) \leq \chi''(K_{n+1}) = n + 1 = \Delta(K_n) + 2$.
- How to get lower bound?
- Counting Technique
- $\alpha_T(K_n) \leq n/2$ and $VE(K_n) = n + (n(n-1)/2 = n(n+1)/2$. So, $\chi''(K_n) \geq n + 1$.

Total Coloring of K_n, n is even

- For even n, $\chi''(K_n) \leq \chi''(K_{n+1}) = n + 1 = \Delta(K_n) + 2$.
- How to get lower bound?
- Counting Technique
- $\alpha_T(K_n) \leq n/2$ and $VE(K_n) = n + (n(n-1)/2 = n(n+1)/2$. So, $\chi''(K_n) \geq n + 1$.
- So, $\chi_T(K_n) = n + 1 = \Delta(K_n) + 2$ is n is even. Hence, K_n is Type-II for n even.

Conformability

Definition

Conformable Graph: A $(\Delta(G)+1)$-vertex coloring of G is said to be a conformable coloring if $def(G) \geq (\Delta(G)+1) - r$, where r is the number of color classes with same parity of V, and
$def(G) = \sum_{v \in V(G)} (\Delta(G) - d(v))$.
A graph that admits a conformable vertex coloring is called a conformable graph..
Note: Some of the $V_i's$ can be empty.

- K_n is conformable if n is odd else not conformable.
- $K_{n,n}$ is conformable if n is even.

Necessary Condition for Type I Graph

Theorem

(chetwynd and Hilton(88)) Every Type-I graph is Conformable.

- Suppose π be a $(\Delta + 1)$-total coloring of G using colors $c_1, c_2, \ldots, c_s$. Let n_i be the number of vertices of G colored c_i. Let r of the numbers $n_1, n_2, \ldots, n_s$ are of same parity as $n = V(G)$. The remaining $s - r$ numbers are of opposite parity as n. Let V_i be a color class that is of different parity than V. Then $V - V_i$ is odd and hence there is vertex i $v \in V \setminus V_i$ that is not colored with i and none of its incident edges is colored with i in the total coloring. So, $d(v) < \Delta$. So, each color class (vertex coloring) of different parity than V contributes 1 to the $def(G)$. Hence, $def(G) \geq s - r$. □

Total Coloring of K_{2n} Using non-conformabilty

- $\chi''(K_{2n}) \leq \chi''(K_{2n+1}) = 2n + 1$. So, $\chi''(K_{2n}) \leq \Delta(K_{2n}) + 2$.
- Note that, K_{2n} is not Conformable as in every $2n$- vertex coloring, each color class is a singleton and hence is of different parity than V. For the $2n$-coloring to be conformable, $def(G)$ should have been $2n$. However, $def(G) = 0$. So, $\chi''(K_{2n}) \geq 2n + 1$.
- Hence, K_{2n} is Type-II.

Conformability is NOT a SUFFICIENT CONDITION for TYPE I Graph

- For each even $n \geq 2$, $K_{n,n}$ is conformable. However, $K_{n,n}$ is Type-II.
- **HILTON's Conjecture:** Let $\Delta(G) \geq \frac{1}{2}(n+1)$. Then G is not type 1 iff either G contains a non-conformable subgraph H with $\Delta(H) = \Delta(G)$ or $\Delta(G)$ is even and G contains a subgraph H obtained by inserting a new vertex into an edge of $K_{\Delta(G)+1}$.

Total Coloring of $K_{m,n}$

- $G = (X, Y, E)$, $X = \{x_1, x_2, \ldots, x_m\}$, and $Y = \{y_1, y_2, \ldots, y_n\}$
- Assume that $m \geq n$.Let $A_p = \{y_i x_{p+i} | i = 1, 2, 3, ..., n\}, p = 1, 2, 3, ...m$, the indices are modulo m. A_p is a complete matching.
- Suppose $m > n$. Then $B_p = A_p \cup \{x_p\}$ and $B_{m+1} = Y$. Thus $\chi_T(G) = m + 1$.
- For $m = n$, $\alpha_T(G) \leq m$ and hence $\chi_T(G) \geq m + 2$. Take $A_{m+1} = X, A_{m+2} = Y$. Hence $A_1, A_2, \ldots, A_m + 2$ are the color classes. Hence, $\chi_T(G) = m + 2$.

Strong Evidence for TCC to HOLD TRUE

Let P_n be the probability of graphs of order n having $\chi_T(G) > \Delta(G) + 2$.

1. McDiarmid(90) $P_n \to 0$ as $n \to \infty$
3. Hind: Almost every graph satisfies TCC.

Complexity Status

1. Arroya(89) Determining $\chi_T(G)$ is NP-hard.
2. Sanchez-Arroya: Deciding whether $\chi_T(G) = \Delta(G) + 1$ even for regular bipartite graph is NP-complete.

How to show TCC holds for a class of graphs? Proof Technique I

Lemma (Chetwyand and Hilton; Yap, Wang and Zhang [1986])

If G contains an independent set S such that $|G - S| \leq \Delta + 1$, then

$$\chi''(G) \leq \Delta + 2$$

- Find a maximal matching M in $G - S$. Color the vertices in S and edges in M using the same color.
- Add a dummy vertex v^* to G, and join edges between v^* and vertices in $G - S$. We call such graph G^*.
- If $\Delta(G^*) = \Delta(G) + 1$, then $G^*_{\Delta+1}$ is a forest and then G^* admits a $\Delta + 1$-edge coloring.
- If $\Delta(G^*) = \Delta$, use edge coloring results to give an edge coloring to G^* using $\Delta(G) + 1$ colors.
- Remove v^* and the edges incident with it, and color the vertex w at the other end with the color of the edge v^*w in G^*.

Proof Technique I

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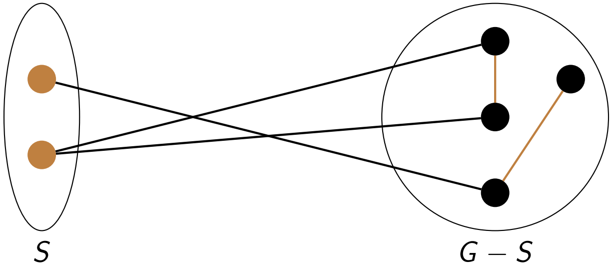

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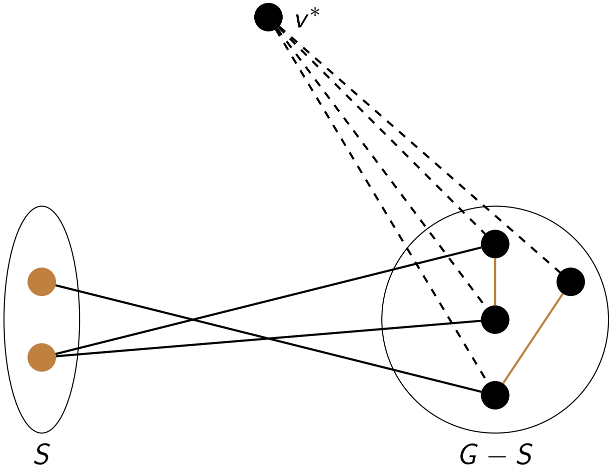

Proof Technique I

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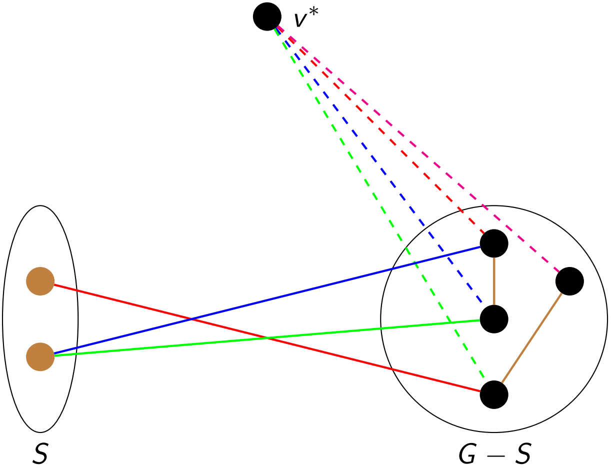
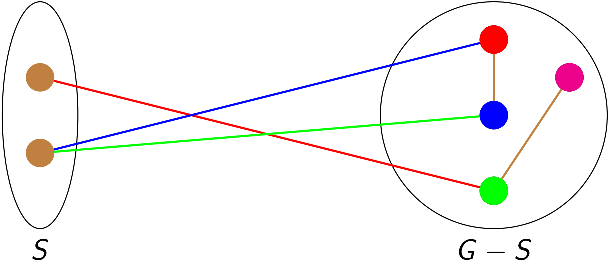


Proof Technique I

- Remove the dummy vertex v^* and the edges incident with it, and color the vertex w at the other end with the color of the edge v^*w in G^*.



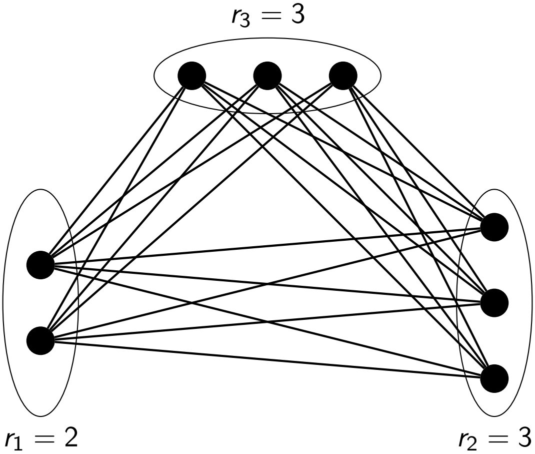
Complete Multipartite Graphs

Definition

The complete p-partite graph $K = K(r_1, \ldots, r_p)$, $r_1 \leq r_2, \cdot, \leq r_p$, is the graph with vertex set $V(K) = \cup_{i=1}^{p} V_i$ with $|V_i| = r_i$for $1 \leq i \leq p$ (each set V_i is called a part) in which two vertices are joined if and only if they occur in different parts of K.

Complete Multipartite Graphs

$K(r_1, r_2, r_3) = K(2, 3, 3)$



Results

Lemma (Chetwyand and Hilton; Yap, Wang and Zhang [1986])

Let G be a graph of order n. If G contains an independent set S of vertices such that $|G - S| \leq \Delta + 1$, then

$$\chi''(G) \leq \Delta + 2$$

Theorem (Yap[89])

For a complete multipartite graph $K(V_1, V_2, \ldots, V_p)$,

$$\chi''(K) \leq \Delta(K) + 2$$

Proof.

Take $S = V_p$, the largest part. Now $|V(K) \setminus S| \leq \Delta(G) + 1$. Hence TCC holds for $K(V_1, V_2, \ldots, V_p)$. □

Limitations of Technique I

The requirement that $|G - S| \leq \Delta(G) + 1$ or in other words that G has an independent set S of size at least $|V(G)| - \Delta(G) - 1$ is **too stringent, and very few graphs satisfy this criteria.**

Proof Technique II

- We look for multiple independent sets $S_1, \ldots, S_k$ and corresponding maximal independent matchings $M_1, \ldots, M_k$ such that
$$|G - \cup_{i=1}^{k} S_i| \leq \Delta(G) + 2 - k.$$
- We need to show the existence of such independent sets $S_1, \ldots, S_k$ and corresponding maximal independent matchings $M_1, \ldots, M_k$. We color vertices of each independent set S_i and corresponding maximal matching M_i with the same color.
- Add a dummy vertex v^* to G, and join edges between v^* and vertices in $G - \cup_{i=1}^{k} S_i$. We call such graph G^*.
- Use edge coloring results to give an edge coloring to G^* using $\Delta(G) - k + 2$ colors.
- We remove the dummy vertex v^* and the edges incident with it, and color the vertex w at the other end with the color of the edge $v^* w$ in G^*.

Limitations

- The requirement that there exists multiple independent sets $S_1, \ldots, S_k$ and corresponding maximal independent matchings $M_1, \ldots, M_k$ such that
$$|G - \cup_{i=1}^{k} S_i| \leq \Delta(G) + 2 - k.$$
is **still too stringent, and few additional graphs satisfy this criteria.**

Total Coloring Problems for some class of graphs

Total Coloring Problem for some class of graphs

Class of Graphs	Conjecture?	Classification Problem
Complete Graphs, K_n	Holds	$\begin{cases} \text{Type 2} & \text{if } n \text{ is even,} \\ \text{Type 1} & \text{if } n \text{ is odd.} \end{cases}$
Complete Bipartite Graphs, $K_{n,m}$	Holds	$\begin{cases} \text{Type 2} & \text{if } n = m, \\ \text{Type 1} & \text{if } n \neq m. \end{cases}$
Complete Multipartite Graphs, K	Holds	$\begin{cases} \text{Type 1} & \text{if } \lvert V(K) \rvert \text{ is odd,} \\ ? & \text{Otherwise.} \end{cases}$
Interval Graphs, G	Holds	$\begin{cases} \text{Type 1} & \text{if } \Delta(G) \text{ is even,} \\ ? & \text{Otherwise.} \end{cases}$
Planar Graphs, G	Holds except $\Delta = 6$	$\begin{cases} \text{Type 1} & \text{if } \Delta(G) \geq 9, \\ ? & \text{Otherwise.} \end{cases}$

Total Coloring Conjecture under specific conditions

Graphs of low degree

Theorem (Rosenfeld [1971] and Vijayaditya[1971])

For any graph G having $\Delta(G) = 3$,

$$\chi''(G) \leq \Delta(G) + 2$$

Theorem (Kostochka[1977])

For any graph G having $\Delta(G) = 4$,

$$\chi''(G) \leq \Delta(G) + 2$$

Theorem (Kostochka[1978])

For any graph G having $\Delta(G) = 5$,

$$\chi''(G) \leq \Delta(G) + 2$$

Graphs of high degree

Theorem (Yap, Wang and Zhang[1989])

For any graph G of order n having $\Delta(G) \geq n-4$,

$$\chi''(G) \leq \Delta(G) + 2$$

Theorem (Yap and Chew[1992])

For any graph G of order n having $\Delta(G) \geq n-5$,

$$\chi''(G) \leq \Delta(G) + 2$$

Theorem (Hilton and Hind[1993])

For any graph G of order n having $\Delta(G) \geq \frac{3}{4}|V(G)|$,

$$\chi''(G) \leq \Delta(G) + 2$$

Graphs of high degree

Theorem (Xie and Yang[2002)

Let $G \neq K_2$ be a graph of even order. If $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| - \frac{5}{2}$, then $\chi''(G) \leq \Delta + 2$.

Theorem (Xie and He[2005)

Let G be a regular graph of even order. If $\delta(G) \geq \frac{2}{3}|V(G)| + \frac{23}{6}$, then $\chi''(G) \leq \Delta + 2$.

Some Results on Classification Problem

Theorem (Hilton [1990])

Let $n \geq 1$, let J be a subgraph of K_{2n}, let $e = |E(J)|$ and let j be the maximum size (i.e. number of edges) of a matching in J. Then

$$X''(K_{2n} - E(J)) = 2n + 1$$

if and only if $e + j \leq n - 1$.

Theorem (Chen and Fu[1992])

Let G be a graph of order $2n$ and $\Delta(G) = 2n - 2$. Then G is of Type 2 if and only if G^c is a disjoint union of an edge and a star having $2n - 3$ edges.

Theorem (Xie and Yang[2002])

Let $G \neq K_2$ be a graph of even order and G_Δ be a forest. If $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| - \frac{3}{2}$, then $\chi''(G) = \Delta + 1$.

Classification: Complete Multipartite graph with odd Order

Theorem (Chew and Yap[92]; Hoffman and Rodger[92])

Let K be a complete multipartite graph such that $|V(K)|$ is odd,

$$\chi''(K) = \Delta(K) + 1$$

Classification: Complete Multipartite graph with even Order

Theorem (Hoffman and Rodger[96])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)| = 2n$. If

$$def(K) \geq \begin{cases} 2n - r_1 & \text{if } p = 2 \text{ or} \\ & \text{if } p \text{ is even, } r_1 \text{is odd, and } r_1 = r_{p-1}, \\ 2n - r_p & \text{otherwise,} \end{cases}$$

then K is of Type 1, where $def(G) = \Sigma_{v \in V(G)}(\Delta(G) - d_G(v))$.

Conjecture

Conjecture (Hoffman and Rodger[96])

A complete multipartite graph $K(r_1, r_2, \ldots, r_p)$ is of Type 2 if and only if

1. $p = 2$ and K is regular, or
2. $|V(K)|$ is even and $def(K)$ is less than the number of parts of odd size.

Results based on size of parts

Theorem (Chew and Yap[92])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph. If $r_1 < r_2$, then K is of Type 1.

Theorem (Dong and Yap[2000])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)|$ is even. If $r_2 \leq r_3 - 2$, then K is of Type 1.

Theorem (Dalal, Panda and Rodger[2016])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)|$ is even. If $r_2 < r_3$, then K is of Type 1.

Theorem (Dalal, Panda and Rodger[2023])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)|$ is even. If $r_3 < r_4$, then K is of Type 1.

Theorem (Dalal, Panda [2024])

Let $K[V_1, V_2, \ldots, V_p]$ be a complete multipartite graph of even order such that $|V_1| = \ldots = |V_k| = r < |V_{k+1}| \leq \ldots |V_p|$. If,

$$\frac{p}{k} \geq \begin{cases} 4 & \text{when } r \text{ is even,} \\ 7 & \text{when } r \text{ is odd.} \end{cases}$$

then K is of Type 1.

Results based on numbers of parts

Theorem (Chew and Yap[92])

Let $K(r_1, r_2, \ldots, r_3)$ be a complete 3-partite graph of even order. Then K is of Type 1.

Theorem (Dong and Yap[2000])

Let $K(r_1, r_2, \ldots, r_4)$ be a complete 4-partite graph of even order. Then K is of Type 2 if and only if def(K) is less than the number of parts of odd size.

Theorem (Dalal and Rodger[2014])

Let $K(r_1, r_2, \ldots, r_5)$ be a complete 5-partite graph of even order. Then K is of Type 2 if and only if def(K) is less than the number of parts of odd size.

Theorem (Dalal and Panda[2024])

Let $K(r_1, r_2, \ldots, r_6)$ be a complete 5-partite graph of even order. Then K is of Type 2 if and only if def(K) is less than the number of parts of odd size.

Clasification of high degree graphs

Chew (Discrete Math, 1999) proved the following theorem for graphs of odd order and high degree:

Theorem

If G is a graph of odd order, minimum degree $\delta(G)$ such that $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| + |V(G_\Delta)| + \frac{5}{2}$, then $\chi''(G) = \Delta(G) + 1$.

However, such sufficient conditions for graphs of even order are not known. The classification problems for graphs of even order is generally believed to be more difficult. Accordingly, graphs of even order have been studied under certain constraints. Other than the classification result for graphs of even order with $\Delta = |V(G)| - 1$ by Hilton and $\Delta = |V(G)| - 2$ by Chen and Fu, the only result that the authors are aware for general graphs is the following result in 2003 by Xie and Yang.

Theorem

(Xie and Yang, Discrete Mathematics, 2003) Let $G \neq K_2$ be graph of even order and G_Δ be a forest. If $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| - \frac{3}{2}$, then $\chi''(G) = \Delta(G) + 1$.

The above Theorem has been used to prove many results. However, it puts a stringent condition on G_Δ to be a forest which limits its applicability.

More precisely, we prove that

Theorem

Let $G \neq K_2$ be a graph of even order such that G_Δ is triangle-free. Then, $\chi''(G) = \Delta(G) + 1$ if

1. $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| - \frac{3}{2}$ *when* $|G_\Delta| \leq |V(G)| - \Delta(G) + 1$,
2. $\delta(G) + \frac{1}{2}\Delta(G) \geq |V(G)| + |G_\Delta| + \frac{1}{2}$, *otherwise.*

The following corollary of Theorem 42 is a generalization of the Theorem 41 of Xie and Yang, as a graph G whose core is a forest satisfies the hypothesis of the corollary.

Corollary

Let G be a graph of even order such that G_Δ is triangle-free and has a pendant vertex. If $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| - \frac{3}{2}$, then $\chi''(G) = \Delta(G) + 1$.

Proof of Corollary 43.

Since G_Δ has a pendent vertex, say v, there are $\Delta - 1$ vertices in G in $V(G) \setminus V(G_\Delta)$ adjacent to v. This implies $(\Delta - 1) + |V(G_\Delta)| \leq |V(G)|$. Thus, $|V(G_\Delta)| \leq |V(G)| - \Delta + 1$. By Theorem 42, G is of Type 1. $\square$

We also prove the following theorem which provides sufficient conditions for graphs with triangle-free to be of Type 1.

Theorem

Let $G \neq K_2$ be a graph such that G_Δ is triangle-free. If

$$\delta(G) + 3\Delta(G) \geq \frac{7}{2}|V(G)| + \frac{5}{2}.$$

then $\chi''(G) = \Delta(G) + 1$.

Problems in Total Coloring

1. Upper Bounds for Total Coloring
2. Settling the Total Coloring Conjecture under specific conditions.
3. Settling the Total Coloring Conjecture for class of graphs.
4. Solve the classification problem for class of graphs for which Total Coloring Conjecture holds.

Upper Bounds for Total Coloring

Probabilistic Method

Theorem (Hind[1990])

For any graph G ,

$$\chi''(G) \leq \Delta(G) + \lceil \log |V(G)| \rceil + 3$$

Proof.

- Wlg ssume $n \geq 3$. let $l = \lceil \log |V(G)| \rceil + 2$. Consider an arbitrary $\Delta + 1$ vertex coloring $C = \{S_1, S_2, \ldots, S_{\Delta+1}\}$ and an arbitrary $\Delta + 1$ edge coloring $D = \{M_1, M_2, \ldots, D_{\Delta+1}\}$ of G.
- Let $C_1, C_2, \ldots, C_{(\Delta+1)!}$ be the $(\Delta + 1)!$ vertex colourings which are obtained by permuting the colour class names of C.
- We show that for some i, combining C_i with D yields a reject graph R with $\Delta(R) \leq l - 1$ ($R_i = \cup_{i=1}^{\Delta+1} \{xy | xy \in M_j$ and x or y receives colour j under $C_i\}$.

$\square$

- Thus, R can be colored with l new colors, thereby completing the desired $\Delta + l + 1$ total colouring of G.

proof Contd...

- To do so, we consider picking a C_i uniformly at random and let $R = R_i$ be the random reject graph thereby obtained.
- We show that the expected number of vertices of degree at least l in R is less than one and thereby prove that there exists an R_i with maximum degree less than l.
- By the Linearity of Expectation, to show that the expected number of vertices of degree at least l is less than 1, it is enough to show that for each vertex v, $Pr(d_R(v) \geq l) < \frac{1}{n}$.
- Now, at most one edge incident to v is in R because it conflicts with v. So we consider the event that there are $l - 1$ edges incident to v which conflict with their other endpoint. We need only show that the probability of this event is less than $\frac{1}{n}$.

proof Contd..

- We actually show that for any vertex v, the expected number of sets of $l-1$ edges incident to v, all of which are in R because they conflict with their other endpoint is less than $\frac{1}{n}$. Applying Markovs Inequality, ($P(X \geq a) \leq \frac{E(X)}{a}$, $a > 0$) we obtain the desired result.
- To this end, we first compute the probability that a particular set $\{vu_1, vu_2, \ldots, vu_{l-1}\}$ of $l-1$ edges incident to v are all in R because they conflict with their other endpoint. We let α_i be the colour of vu_i. We let β_i be the colour that u_i is assigned under C. We are computing the probability that our random permutation takes β_i to α_i for $1 \leq i \leq l-1$.
- This probability is zero if the β_i are not distinct.

proof Contd...

- Otherwise, the probability that the permutation does indeed take each of the $l-1$ colors β_i to the corresponding α_i is $\frac{(\Delta+1-(l-1))!}{(\Delta+1)!}$.
- Now, there are at most $\binom{\Delta}{l-1}$ sets of $l-1$ edges incident to v in G. So the expected number of sets of $l-1$ edges incident with v which conflict with their other endpoint is at most:
- $\binom{\Delta}{l-1}\left(\frac{\Delta+1-(l-1))!}{(\Delta+1)!}\right. < \frac{1}{(l-1)!}$.
- It is easy to see that $(\lceil \log n \rceil + 1)!$ is greater than n provided n is at least three, so the result holds. $\square$

Lovasz Local Lemma and bounds

Theorem

The Lovasz Local Lemma: Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A \in \mathcal{E}$,

- a $Pr(A) = p < 1$, *and*
- b *A is mutually independent of a set of all but at most d of the other events. If $4pd \leq 1$ then with positive probability, none of the events in $\mathcal{E}$ occur.*

- assign each vertex a uniformly random colour without considering the colours assigned to the other vertices.
- Our bad events would each be determined only by the colours on a cluster of vertices which are all very close together, and so events corresponding to clusters in distant parts of the graph would occur independently.
- The problem with this approach is that it is very unlikely to generate a proper vertex colouring.
- To overcome this problem, consider a two phase procedure, consisting of a random initial phase which retains the flavour of the random procedure, followed by a deterministic phase which ensures that we have a proper total colouring.

- We first randomly partition V into k sets $V_1, V_2, \ldots, V_k$ such that for each i, the graph H_i induced by V_i has maximum degree at most $l-1$ with l near $\frac{\Delta}{k}$.
- We then greedily color the vertices of each H_i using the colors in $C_i = \{(i-1)l, \ldots, il-1\}$. This yields a kl coloring of $V(G)$.

- We fix any $\Delta + 1$ edge colouring $\{M_1, \dots, M_{\Delta+1}\}$ before performing this process.
- An edge xy conflicts with the endpoint x if $xy$is colored with a colour in C_i and x is assigned to V_i. We note that if e does not conflict with x then in the second phase, the color assigned to x will be different from that used on e.
- The advantage to widening our definition of conflict in this way is that now the conflicts depend only on the random phase of the procedure, and this allows us to apply the Local Lemma

Theorem

For any graph G with maximum degree Δ sufficiently large, $\chi^{''}(G) \leq \Delta + \Delta^{\frac{3}{4}}$.

proof

As usual, we can assume that G is Δ-regular. Set $k = k_\Delta = \lceil \Delta^{\frac{1}{3}} \rceil$, $l = l_\Delta = \lfloor \frac{\Delta + \Delta^{\frac{3}{4}}}{k} \rfloor$. We fix an arbitrary $\Delta + 1$-edge colouring of G using the colours $1, 2, \ldots, \Delta + 1$. We then specify a vertex colouring of G using the colours $0, 1, \ldots, kl - 1 = \Delta + \Delta^{\frac{3}{4}} - 1$ as follows.

We first partition $V(G)$ into $V_1, V_2, \ldots, V_k$ such that

(i) fore ach vertex v and part i, $|N_v \cap V_i| \leq l - 1$, and

(ii) For each vertex v, there are at most $\Delta^{\frac{3}{4}} - 3$ edges$e = uv$ such that $u \in V_i$ and e has a colour in C_i.

Our next step will be to refine this partition into a proper colouring, colouring the vertices of V_i using the colours in C_i.

By (i), we can do so using the simple greedy procedure since the subgraph induced by V_i has maximum degree $l - 1$. By (ii), the reject graph formed has maximum degree at most $\Delta^{\frac{3}{4}} - 2$ (there is a 2 and not a 3 here because we may reject an edge incident to v because it has the same colour as v). Recolouring these edges with at most $\Delta^{\frac{3}{4}} - 1$new colours yields the desired total colouring of G.

It only remains to show that we can actually partition the vertices so that (i) and (ii) hold. To do so, we simply assign each vertex to a uniformly random part (where of course, these choices are made independently). For each v, i, we let $A_{v,i}$ be the event that (i) fails to hold for $\{v, i\}$ and B_v be the event that (ii) fails to hold for v. We will use the Local Lemma to prove that with positive probability none of these bad events occur. B_v and $A_{v,i}$ are determined by the colours of the vertices adjacent to v. Thus, by the Mutual Independence Principal, they are mutually independent of all events concerning vertices which are at distance more than 2 from v, and so every event is mutually independent of all but at most $(k+1)\Delta^2 < \Delta^3$. other events. We will show that the probability that any particular bad event holds is much less than $\frac{1}{4\Delta^3}$. Thus, by the Local Lemma, there exists a colouring satisfying (i) and (ii).

Consider first the event B_v. Let Rej_v be the set of edges $e = uv$ with the property that e has a colour in C_i and $u \in V_i$. Since there are k parts, the probability that this occurs for a given e is exactly $\frac{1}{k}$. Furthermore, as the choices of the parts are independent, the size of Rej_v is just the sum of Δ independent $0 - 1$ variables each of which is 1 with probability $p = \frac{1}{k}$Â·
Applying the Chernoff Bound for $BIN(\Delta, p)$ we obtain:
$Pr(||Rej_v| - \frac{\Delta}{k}| > \frac{\Delta}{k}) \leq 2e^{-\frac{\Delta}{3k}}$,
Since $k = \lceil \Delta^{\frac{1}{3}} \rceil$ and $\frac{\Delta^{\frac{3}{4}}}{2} > \frac{\Delta}{k}$, it follows that for Δ sufficiently large,
$Pr(B_v) \leq 2e^{-\Delta^{\frac{1}{2}}}$.

The size of $N_v \cap V_i$ is just the sum of Δ independent $0-1$ variables each of which is 1 with probability $\frac{1}{k}$, and so applying the Chernoff Bound as above we obtain that for large Δ,
$Pr(A_{v,i}) \leq Pr(||N_v \cap V_i| - \frac{\Delta}{k}| > \frac{\Delta^{\frac{3}{4}}}{2}) \leq 2e^{-\Delta^{\frac{1}{2}}}$.

Total Coloring

- First Approach: Choose an edge colouring and then choosing a vertex colouring which didn't significantly conflict with it. We then obtained a total colouring by modifying the edge colouring so as to eliminate the conflicts.
  The opposite approach: first choosing a vertex colouring and then choosing an edge colouring which does not conflict at all with the vertex colouring, thereby obtaining a total colouring.
- It is believed that (If List Coloring Conjecture: $\chi(G) = \chi_L(G)$ for a Line graph) is True) for every $\Delta + 3$ -vertex coloring, there is some edge colouring using the same colours with which it does not conflict.
- We note that the analogous statement does not hold for edge colouring, even if we replace the $\Delta + 3$ by $2\Delta - 1$.
- To see this, consider the graph obtained from a clique of order Δ by adding a pendant edge at each vertex The clique has maximum degree $\Delta - 1$ and hence has a Δ edge coloring.

- We can extend this to a $\Delta + 1$-edge coloring of the whole graph by coloring the pendant edge from a vertex v of the clique with the color which does not appear on any of the other edges incident with v. But now Δ new colors are needed to colour the vertices of the clique if we are to avoid conflicts.

- Given that we cannot always extend $\Delta + 3$ -edge colourings without introducing new colours, why should we expect to be able to extend $\Delta + 3$ - vertex colourings?
- The answer is that colouring the vertices places very few restrictions on the edge colouring. Specifically, consider fixing a vertex colouring C which uses the colours $\{1, 2, \ldots, \Delta + 3\}$.
- Then, for each edge e there is a list of $\Delta + 1$ acceptable colours whose assignment to e will not generate a conflict.

- Thus, if the List Colouring Conjecture is correct there is an edge colouring in which each edge receives an acceptable colour and hence which does not conflict with C.
- We note that $\Delta + 3$ is best possible here, as Hind has given examples of $\Delta + 2$-vertex colourings which cannot be extended to $\Delta + 2$ -total colourings.
- Although it is believed that every $\Delta + 3$ -vertex colouring can be extended to a $\Delta + 3$-total colouring, vertex colourings with a special property are considered which makes them easier to extend.

Definition A k-frugal colouring is a proper vertex colouring in which no colour appears more than k times in any one neighbourhood.
Hind, Molloy and Reed [80] proved the following:

Theorem

Every graph G with maximum degree Δ sufficiently large, has a $\log^8 \Delta$-frugal $\Delta + 1$-colouring.

Theorem

There exists a Δ_0 *such that for* $\Delta \geq \Delta_0$, *every log*$^8\Delta$*-frugal* $\Delta + 1$*-colouring of a graph* G *with maximum degree* Δ *can be extended to a* $\Delta + 2\log^{10}\Delta + 2$ *-total colouring o* G.

Combining these two results yields:

Theorem

If G *has maximum degree* Δ, *the* $\chi''(G) \leq \Delta + O(\log^{10}\Delta)$.

Some Results

Theorem (Hind[1990])

For any graph G and any positive integer $k \leq \Delta(G)$,

$$\chi''(G) \leq \chi'(G) + \lceil \frac{\chi(G)}{k} \rceil + k$$

bounds...

Theorem (Chetwynd and Haggkvist[1992])

If G is a graph of order n and k is an integer such that $k! \geq n$,

$$\chi''(G) \leq \chi'(G) + k$$

Theorem (Sanchez and Arroyo[1995])

For any graph G,

$$\chi''(G) \leq \chi'(G) + \lfloor \frac{\chi(G)}{3} \rfloor + 2$$

Some Bounds which depend only on maximum degree

Theorem (Koshtochka[1977])

For any multigraph G with $\Delta(G) \geq 6$,

$$\chi''(G) \leq \frac{3}{2}\Delta(G)$$

Theorem (Molly and Reed[1998])

For any graph G,

$$\chi''(G) \leq \Delta(G) + 10^{26}$$

Remark. Molloy and Reed have mentioned that with much more effort i10^{26} can be brought down to 500.

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Thank You