On Conflict-free Coloring of Graphs

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Conflict-free Coloring

Definition

Given a graph G, a conflict-free coloring is an assignment of colors to $V(G)$ such that

- For each vertex v there is at least one color appearing exactly once in the neighborhood of v.

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We consider both closed and open neighborhoods.

Conflict-free closed neighborhood (CF-CN) coloring

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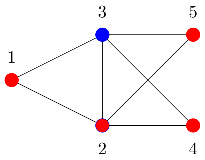

Figure 1: $\chi_{CN}(G) = 2$

Conflict-free open neighborhood (CF-ON) coloring

Definition

Given a graph G, a conflict-free open neighborhood coloring is an assignment of colors to $V(G)$ such that

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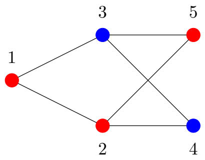

Figure 2: $\chi_{ON}(G) = 2$

CFCN (or) CF-ON problem

Input: A graph G

Question: Find $\chi_{CN}(\mathsf{G})$ (or) $\chi_{ON}(\mathsf{G})$.

Examples: Complete Graphs

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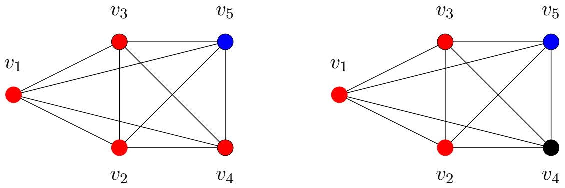

Figure 3: $\chi_{CN}(K_5) = 2$ and $\chi_{ON}(K_5) = 3$

Relation between $\chi(\mathsf{G})$, $\chi_{\mathrm{CN}}(\mathsf{G})$ and $\chi_{\mathrm{ON}}(\mathsf{G})$

- $\chi_{\mathrm{CN}}(\mathsf{G}) \leqslant \chi(\mathsf{G})$ for any graph G.

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- $\chi(G)$ vs $\chi_{ON}(G)$
  - Let K_n' be the graph obtained from the complete graph K_n on vertices by subdividing each edge.
  - $\chi(K_n') = 2$ as it is bipartite.
  - Each of the new vertex of K_n' has degree 2, so all original vertices must receive different colors, hence $\chi_{ON}(K_n') = n$.

Motivation: Frequency Assignment Problem

1. The conflict-free coloring problem was introduced by Even et al. [FOCS 2002] in a geometric setting to study the frequency assignment problem for cellular networks.
2. The cellular networks contains two different types of nodes: base-stations and clients.
3. Fixed frequencies are assigned to base stations to allow connections to clients.
4. The frequency assignment problem on cellular networks is an assignment of frequencies to base stations such that for each client there exists a base station of unique frequency within his region.

Related Work

- Due to both its practical motivations and its theoretical interest, conflict-free coloring has been investigated in several papers.

- Pach and Tardos [Combin. Probab. Comput 2009] initiated the theoretic study of the CF-chromatic number in general graphs and hypergraphs.

- The problem of determining the CF-CN (resp. CF-ON) chromatic number of a graph is NP-complete [Gargano and Rescigno TCS 2015].

- Determining the CF-ON chromatic number is NP-complete even on bipartite graphs.

Restricted Graph Classes [1]

- Interval Graphs
- Split Graphs
- Cographs

[1] I.V., Theoretical Computer Science 2018

Interval Graphs

Definition

A graph $G = (V, E)$ is an *interval graph* if there exists a set $\mathcal{I}$ of intervals on the real line such that there is a bijection $f : V \longrightarrow \mathcal{I}$ satisfying the following: $v_1 v_2 \in E(G)$ if and only if $f(v_1) \cap f(v_2) \neq \emptyset$.

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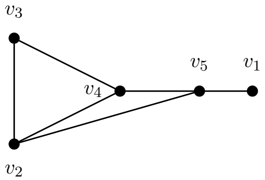
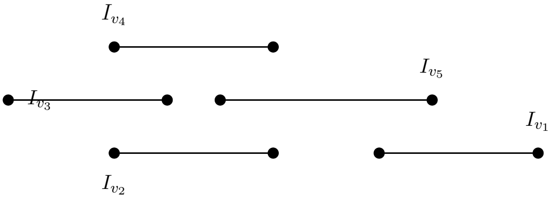




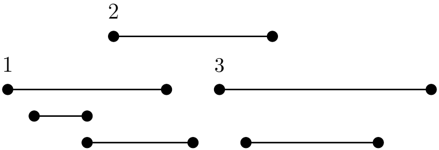
Interval Graphs

Theorem

Let G *be an interval graph with at least one edge, then*

$$\chi_{\mathrm{CN}}(\mathsf{G}) \leqslant 4 \text{ and } \chi_{\mathrm{ON}}(\mathsf{G}) \leqslant 4$$

Idea:



Interval Graphs: Improved bounds

- Fekete and Keldenich [ISAAC-2017] showed that $\chi_{CN}[G] \leqslant 3$, which is tight.

- Bhyravarapu et al.[MFCS-2022] showed that $\chi_{ON}(G) \leqslant 3$, which is tight.

Polynomial time algorithms: Interval Graphs

- Recently Gonzalez and Mann [DAM-2024] showed that CF-CN and CF-ON problems can be solved in polynomial time when mim-width and number of colors are bounded. This includes the class of interval graphs.

- Independent of Gonzalez and Mann [DAM-2024], Bhyravarapu et al. [MFCS-2022] also showed that the CF-CN and CF-ON problems can be solved in polynomial time.

- Their algorithm is more explicit runs in $O(n^5)$ and based on the idea of multi-chain ordering of interval graphs.

Split Graphs

Definition (Split Graph)

A graph $G = (C, I)$ is a split graph if its vertices can be partitioned into a clique and an independent set.

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Theorem

Let $G = (C, I)$ *be a split graph, then*

$$\chi_{CN}[G] \leqslant 3$$

Idea:

- Color one vertex of C with 1 and all remaining vertices (if any) with 2.
- Color all vertices (if any) of independent set I with 3.

Split Graphs: CFCN Coloring

Theorem

Let $G = (C, I)$ *be a split graph without universal vertices.*

1. *If* $|C| = 2$ *then* $\chi_{CN}[G] = 2$.
2. *If* $|C| > 2$ *then* $\chi_{CN}[G] = 2$ *if and only if* $|N(v) \cap I| = 1$ *for every* $v \in C$.

Split Graphs: CFCN Coloring

Theorem

Let $G = (C, I)$ *be a split graph without universal vertices.*

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Theorem

The CF-CN coloring problem is polynomial time solvable on the class of split graphs.

Split Graphs-CFON

Theorem

The CF-ON coloring problem is NP-complete on the class of split graphs.

Cographs

Definition

A graph G is a cograph if G does not contain any path with four vertices as an induced subgraph.

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Definition (Recursive)

The class of cographs can also be defined recursively as follows.

- The one vertex graph K_1 is a cograph.
- Let G_1 and G_2 be two cographs. Then, the disjoint union $G_1 + G_2$ is a cograph.
- Let G_1 and G_2 be two cographs. Then, the join $G_1 \oplus G_2$ is a cograph.

Cographs

Theorem

Let G be a connected co-graph on $n(\geqslant 5)$ vertices , then

$$2 \leqslant \chi_{CN}[G], \chi_{ON}[G] \leqslant 3$$

Idea: Let $G = G_1 \oplus G_2$,

- For CF-CN/CF-ON coloring color one vertex of G_1 with 1, one vertex of G_2 with 2 and all other remaining vertices (if any) of G with 3.

Polynomial-time Algorithm for CF-CN and CF-ON coloring on Cographs

Lemma

Let G *be a cograph with at least five vertices. The graph* G *has a universal vertex if and only if* $\chi_{\mathsf{CN}}(\mathsf{G}) = 2$.

Polynomial-time Algorithm for CF-CN and CF-ON coloring on Cographs

Lemma

Let G be a cograph with at least five vertices. The graph G has a universal vertex if and only if $\chi_{\mathsf{CN}}(\mathsf{G}) = 2$.

Lemma

CF-ON coloring can be solved in polynomial time on cographs.

Proof follows from the following facts:

- Cographs have clique-width at most two.
- CF-ON coloring [IWOCA 2021] can be solved in time $O(2^{O(wk)}n^{O(1)})$, where w is the clique-width of G, k is the number of colors and n is the number of vertices of G.

Open Question: Chordal Graphs

1. Chordal graphs are superclass of interval graphs and split graphs.
2. CF-ON coloring is NP-complete on chordal graphs.

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1. Chordal graphs are superclass of interval graphs and split graphs.
2. CF-ON coloring is NP-complete on chordal graphs.
3. What is the complexity of CF-CN coloring on chordal graphs?

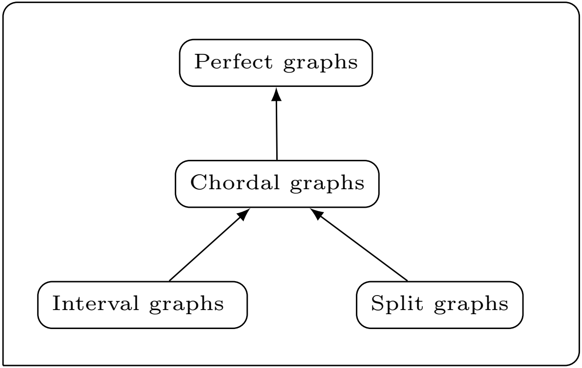
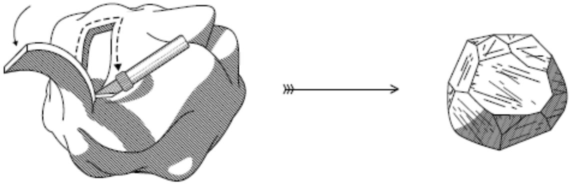


Figure 4: $\longrightarrow$ represents $\subset$ relation.

Parameterized Complexity

- The time complexity of an algorithm is measured not just in terms of the input size but also a additional secondary measurement called parameter.
- The goal is to identify interesting parameterizations of hard problems where we can design algorithms running in time $f(k)poly(n)$.
- Such algorithms are called "fixed-parameter tractable" (FPT) algorithms

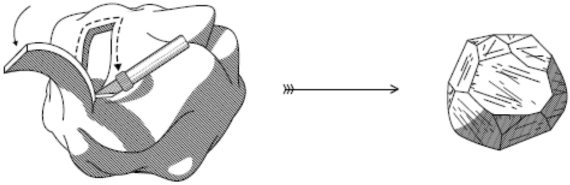
Kernelization



A kernelization for a parameterized problem L is an algorithm that takes an instance (x, k) and maps it in time polynomial in $|(x, k)|$ to an instance (x', k') such that

- $(x, k) \in L \Leftrightarrow (x', k') \in L$
- $|(x', k')| \leqslant f(k)$ where f is a function we call the size of the kernel.

Kernelization



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Theorem

A parameterized problem Π is FPT if and only of it admits a kernel.

How to choose a parameter

In general there are two main approaches in selecting a parameter for graph problems

- The natural parameter is size of solution.
- The parameters which do not involve objective function, which are selected based on structure of the graph called structural graph parameters.

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- In this talk, we concentrate on structural parameters: Vertex cover and distance to cluster

Relation between Parameters

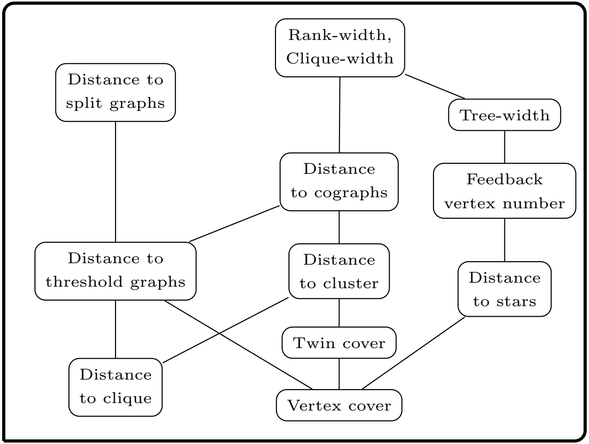

Figure 5: There is a line between two parameters if the parameter below is larger than the parameter above.

Parameterized Algorithms

- Vertex Cover(warmup)
- Distance to cluster

Vertex Cover

Definition

A vertex cover of a graph G is a subset $X \subseteq V(G)$ of vertices such that for every edge of G at least one of its endpoints is in X. A minimum vertex cover is a vertex cover having the smallest possible number of vertices.

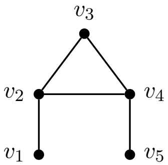

Figure 6: $\{v_2, v_4\}$ is a vertex cover of size two.

Parameterization by Vertex Cover

Input: A graph G, a vertex cover $X \subseteq V(G)$ and an integer k

Parameter: $d := |X|$

Question: Does G have a CF-CN coloring with at most k colors?

Parameterization by Vertex Cover

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Parameter: $d := |X|$

Question: Does G have a CF-CN coloring with at most k colors?

Goal: Design an algorithm with running time $f(d)\mathrm{poly}(n)$

Parameterization by Vertex Cover[2]

Theorem

If G *has a vertex cover of size* d *then*

$$\chi_{CN}[G] \leqslant d+1 \quad \textit{and} \quad \chi_{ON}(G) \leqslant 2d+1$$

[2] Gargano and Rescigno, 2015

Parameterization by Vertex Cover[2]

Theorem

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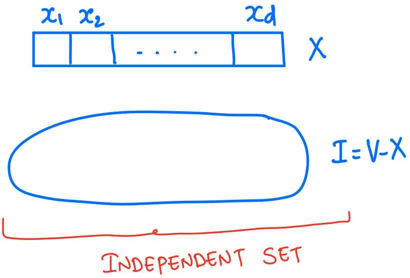
Idea (i):

- Let X be a vertex cover of size d and $I = V - X$ is an independent set.
- To each $u \in I$ assign the color 0.
- Color vertices of X with d distinct colors.



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Parameterization by Vertex Cover

Idea (ii): A CF-ON-$(2d+1)$-coloring C of G can be obtained as follows.

- To each $u \in I$ assign the color $C(u) = 0$.
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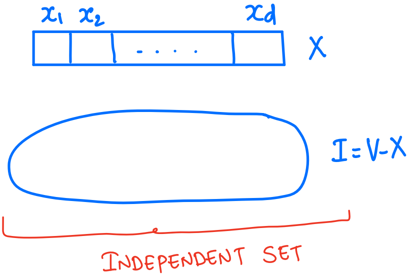
Parameterization by Vertex Cover

Idea (ii): A CF-ON-$(2d+1)$-coloring C of G can be obtained as follows.

- To each $u \in I$ assign the color $C(u) = 0$.
- Color vertices of X with d distinct colors from the set $\{1, \ldots, d\}$.
- For each $x \in X$ if $C(N(x)) = \{0\}$ choose one node $u \in N(x)$ and recolor it with any $C(u) \in \{d+1, ..., 2d\}$ that is not already used by C.



FPT algorithm for CF-CN Coloring w.r.to Vertex Cover

Theorem

CF-CN coloring problem is fixed-parameter tractable when parameterized by the vertex cover number of the input graph.

Proof Idea (ii): Let X be a vertex cover of G of size d.

- Without loss of generality we assume that $k < d + 1$.
- For a subset $Y \subseteq X$, define

$$T_Y = \{x \in I \mid N(x) \cap X = Y\}$$

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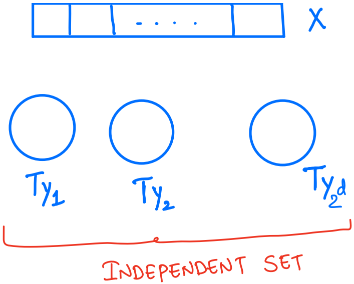
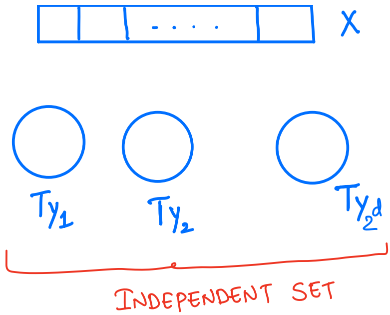


CF-CN Coloring-Vertex Cover



- For each $Y \subseteq X$, if $|T_Y| > d+2$ then removing all vertices except $d+2$ from T_Y does not change $\chi_{CN}(G)$.
- After applying the above rule on input graph G, the number of vertices in the reduced instance is at most $d + (d+2)^{2^d}$.

CF-ON with respect to Vertex Cover

Along the similar lines, we can show that CF-ON is FPT parameterized by vertex cover.

Distance to cluster (or) cluster vertex deletion number

Definition

A cluster graph is a disjoint union of complete graphs.

Definition

The cluster vertex deletion number (or distance to a cluster graph) of a graph G is the minimum number of vertices that have to be deleted from G to get a disjoint union of complete graphs.

Distance to cluster (or) cluster vertex deletion number

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Definition

The cluster vertex deletion number (or distance to a cluster graph) of a graph G is the minimum number of vertices that have to be deleted from G to get a disjoint union of complete graphs.

- The cluster vertex deletion number is an intermediate parameter between the vertex cover number and the clique-width/rank-width.
- We show that both variants of conflict free coloring problems are FPT parameterized by the cluster vertex deletion number.

Cluster vertex deletion number: CFCN

Theorem

If G *has a cluster vertex deletion number* d *then*

$$\chi_{CN}[G] \leqslant d + 2$$

Cluster vertex deletion number: CFCN

Theorem

If G *has a cluster vertex deletion number* d *then*

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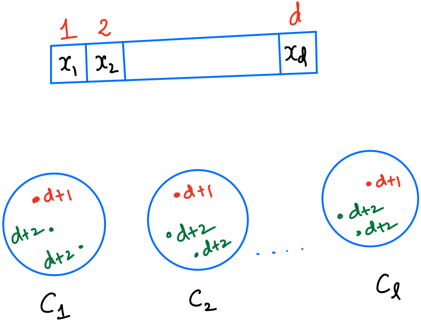
Idea: Let $X \subseteq V(G)$ of size d such that $G - X$ is cluster graph.
A CF-CN $(d+2)$-coloring C_G of G can be obtained as follows.



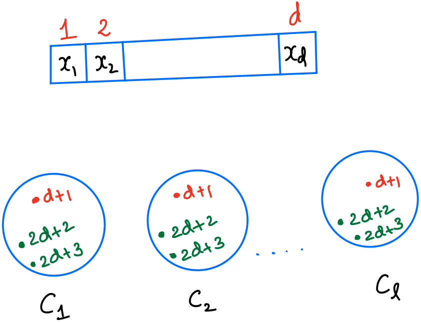
Cluster vertex deletion number: CFON

Theorem

If G *has a cluster vertex deletion number* d *then*

$$\chi_{ON}[G] \leqslant 2d + 3$$

Idea: Let $X \subseteq V(G)$ of size d such that $G - X$ is cluster graph.
A CF-ON $(2d+3)$-coloring C_G of G can be obtained as follows.



Improved Results: cluster vertex deletion number

Bhyravarapu and Kalyanasundaram [WG 2020] improved the bounds and obtained the following results.

Theorem

$\chi_{CN}(G) \leqslant \max\{3, d+1\}$ *and* $\chi_{ON}(G) \leqslant d+3$, *where* d *is cluster vertex deletion number of* G.

FPT Algorithm-CFCN-CVD

Theorem

The CF-CN coloring problem is fixed-parameter tractable when parameterized by the cluster vertex deletion number of the input graph.

FPT Algorithm-CFCN-CVD

Theorem

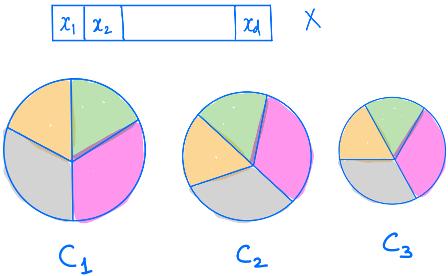
The CF-CN coloring problem is fixed-parameter tractable when parameterized by the cluster vertex deletion number of the input graph.

Idea: Let $X \subseteq V(G)$ of size d such that $G - X$ is cluster graph.

- Without loss of generality we assume that $k < d + 2$.
- For a subset $Y \subseteq X$ and a clique C in $G - X$, define

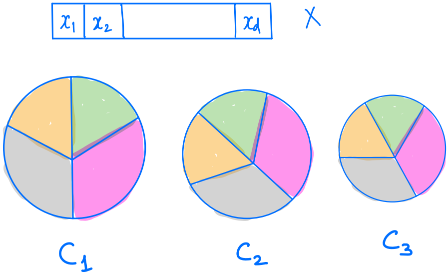
$$T_Y^C = \{x \in C \mid N(x) \cap X = Y\}$$

- This way we can partition vertices of a clique C into at most 2^d subsets (called types), one for each $Y \subseteq X$.



FPT Algorithm-CFCN-CVD

Reduction Rule 1: For a clique $C \in G \setminus X$, if a type $T_Y^C(G)$ has more than $k+1$ vertices for some $Y \subseteq X$, then removing all vertices except $k+1$ from $T_Y^C(G)$ does not change $\chi_{CN}[G]$.

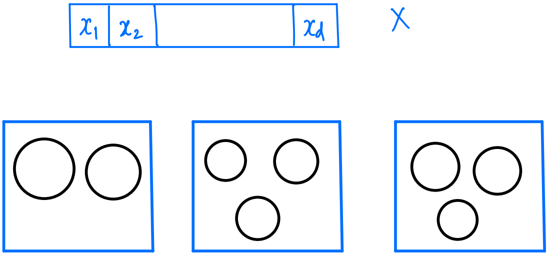


FPT Algorithm-CFCN-CVD

- For each subset $S \subseteq \{0, 1, \ldots, k+1\}^{2^d}$,

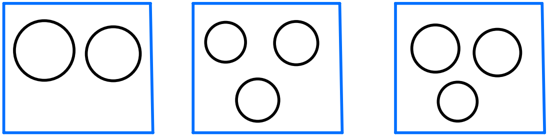
$$T_S(G_1) := \{C \in G_1 \setminus X \mid T^C = S\}$$

- Next, we partition the cliques in $G - X$ based on their type vector.
- This way we can partition cliques of $G_1 \setminus X$ into at most $(k+2)^{2^d}$ subsets (called mega types), one for each $S \subseteq \{0, 1, \cdots, k+1\}^{2^d}$.



FPT Algorithm-CFCN-CVD

Reduction Rule 2: For a subset $S \subseteq \{0, 1, \cdots, k+1\}^{2^d}$, If the mega type $T_S(G)$ has more than $d+1$ cliques then removing all cliques except $d+1$ cliques from $T_S(G)$ does not change the $\chi_{CN}[G]$.



FPT Algorithm-CFCN-CVD

- After applying the above reduction rules on input graph G, the size of the reduced instance is at most $O((k+2)^{2^d}(d+1)(k+1))$.
- As $k < d+2$, we get a kernel of size at most $O(d^{2^d+2})$.

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- As $k < d+2$, we get a kernel of size at most $O(d^{2^d+2})$.

The proof of CF-ON coloring is similar to the CF-CN coloring except some minor changes in the reduction rules.

Other Variants

- Till now, all the vertices of the graph are colored in a conflict-free coloring.
- One variant of conflict-free coloring problem, which asks to color a subset of vertices of the graph maintaining a uniquely colored neighbor for each vertex.

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- One variant of conflict-free coloring problem, which asks to color a subset of vertices of the graph maintaining a uniquely colored neighbor for each vertex.
- Conflict-Free Full Coloring - all the vertices are colored.
- Conflict-Free Partial Coloring - subset of vertices colored.

Other Variants

- Till now, all the vertices of the graph are colored in a conflict-free coloring.
- One variant of conflict-free coloring problem, which asks to color a subset of vertices of the graph maintaining a uniquely colored neighbor for each vertex.
- Conflict-Free Full Coloring - all the vertices are colored.
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Note that this does not change asymptotic results for general graphs: it suffices to introduce one additional color for vertices that are left uncolored.

Partial Conflict-free Coloring

Definition

Given a graph $G = (V, E)$, a partial conflict-free coloring is an assignment of colors to a subset of vertices in V such that every vertex in V has a uniquely colored vertex in its neighborhood.

Notation:

- $\chi^*_{CN}(G)$: for closed neighborhood coloring.
- $\chi^*_{ON}(G)$: for open neighborhood coloring.

Examples

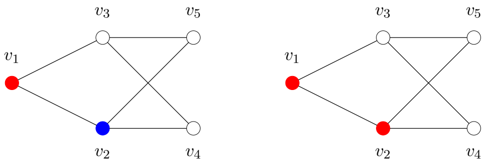

Figure 11: $\chi^*_{CN}(G) = 2$ and $\chi^*_{ON}(G) = 1$

Related work

Abel et al. [SIDMA-2018] studied the problem on planar and outer planar graphs.

- For closed neighborhoods: It is NP-complete to decide whether a planar graph has a conflict-free coloring with one color, while for outerplanar graphs, this can be decided in polynomial time.
- For open neighborhoods: For any planar graph G, $\chi^*_{ON}(G) \leqslant 8$.
- For any outerplanar graph G, $\chi^*_{ON}(G) \leqslant 6$.

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- For any planar graph G, $\chi^*_{ON}(G) \leqslant 6$.
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We have studied the problem(both variants) on restricted graph classes, unit square, unit disk etc.

CF-ON* coloring

This is a joint work with B. Sriram, S. Kalyanasundaram, Tim A. Hartmann and Hung P. Hoang, IWOCA 2021.

Graph Class	**Upper Bound**	**Lower Bound**	**Complexity**
Block graphs	3	3	P
Cographs	2	2	P
Interval graphs	3	3	-
Proper Interval graphs	2	2	-
Unit square	27	3	NP-hard
Unit disk	54	3	NP-hard
Split graphs	-	-	NP-hard

For unit square and unit disk graphs there is still a wide gap between lower and upper bound, and it would be interesting to improve those bounds.

Our Results and Open Questions

- We have shown that both variants (CF-CN* and CF-ON*) of the problem are FPT when parameterized by combined parameters clique-width and number of colors.

- It remains an open question if there exists an FPT algorithm with only clique-width as a parameter.

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- We have shown that both variants (CF-CN* and CF-ON*) of the problem are FPT when parameterized by combined parameters clique-width and number of colors.

- It remains an open question if there exists an FPT algorithm with only clique-width as a parameter.

- Gonzalez and Mann [DAM-2024] showed that both open neighborhood and closed neighborhood variants are polynomial time solvable (XP) when mim-width and the number of colors are bounded.

- As mim-width generalizes clique-width, it is interesting to see if there exists an FPT algorithm parameterized by mim-width and k.

Other Variants

1. Unique maximum coloring [Fabrici et al. DAM 2023]

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4. Odd coloring [Caro et al. Discrete Mathematics 2023]

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Thank you!