ON PROPERTIES OF MODULAR AND DIRECT-CO-DIRECT PRODUCTS

Iztok Peterin

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CALDAM Indo-Slovenia Pre-Conference School on Algorithms and Combinatorics
February 2024

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- A. Kelenc, I.P., Distance formula for direct-co-direct product in the case of disconnected factors, Art Discrete Appl. Math. 6(2) (2023) p2.13 (21p).

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- The distance between u and v is the minimum number $d_G(u, v)$ of edges on a u, v-path in graph G.

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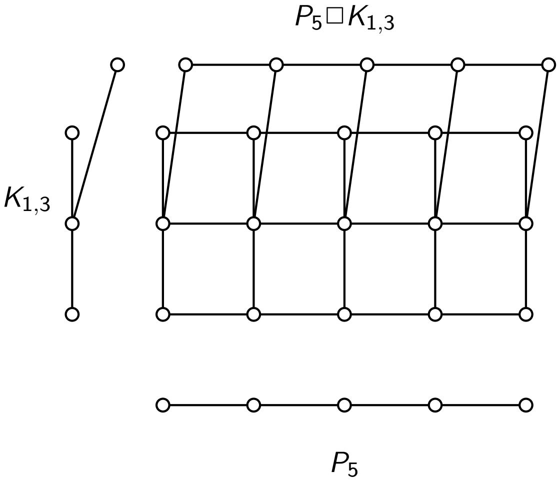
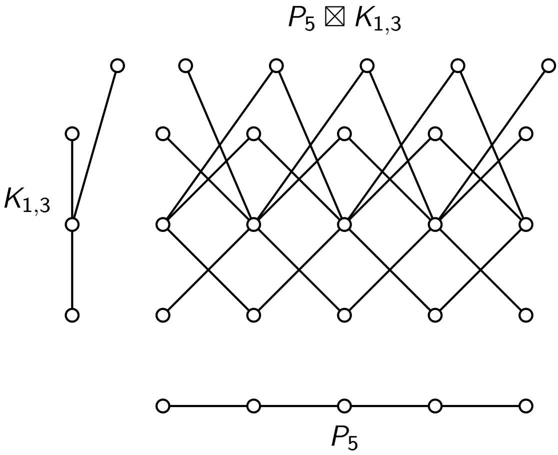
- Modular product and his complementary product.

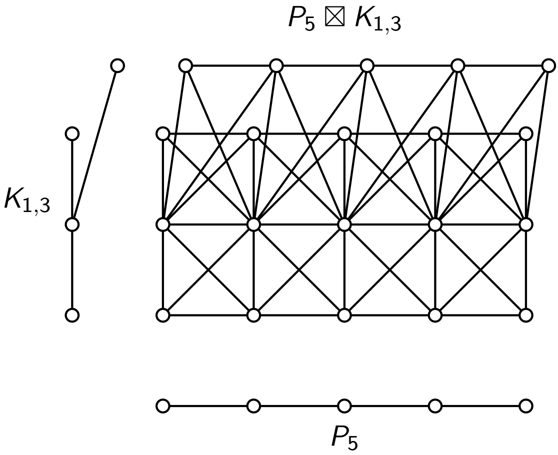
Example of Cartesian product



Examples of direct product



Example of strong product



Distance formulas for products

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$$d_{G\boxtimes H}((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}.$$

Modular product and his complementary product

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- We have $E(G \diamond H) = E(G \square H) \cup E(G \times H) \cup E(\overline{G} \times \overline{H}) = E(G \boxtimes H) \cup E(\overline{G} \times \overline{H})$.

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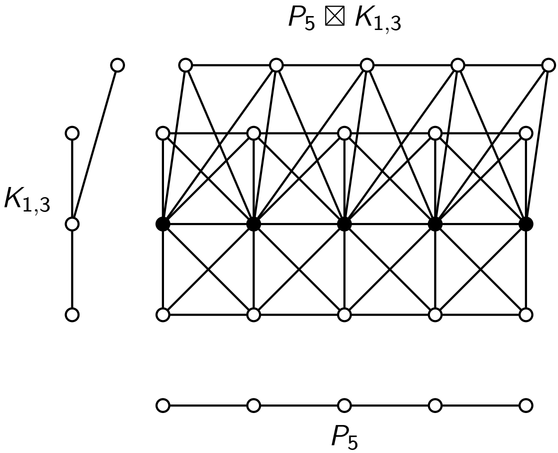
Theorem

The modular product $G \diamond H$ is disconnected if and only if one factor is complete and the other is disconnected or both factors are disjoint union of two complete graphs.

From strong product $K_{1,3} \boxtimes P_3$ to modular product $K_{1,3} \diamond P_3$



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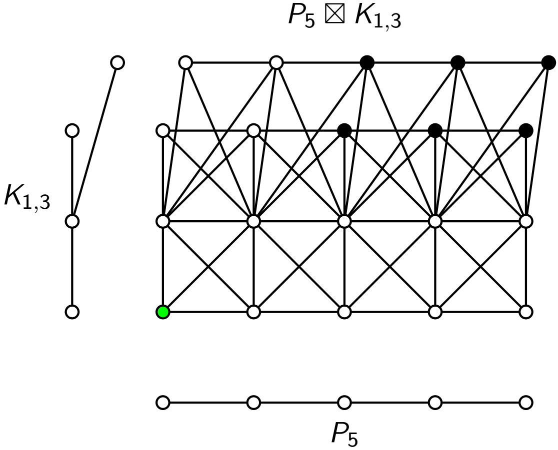

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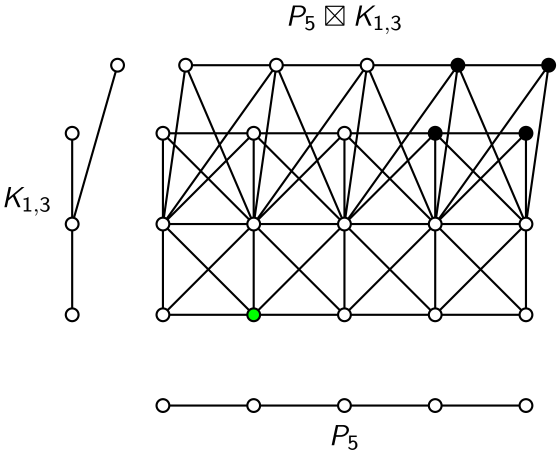

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- $E(G \circledast H) = E(G \times H) \cup E(\overline{G} \times \overline{H}) = E(G \diamond H) - E(G \Box H)$.
- Many graph products can be expressed with the introduced graph products.

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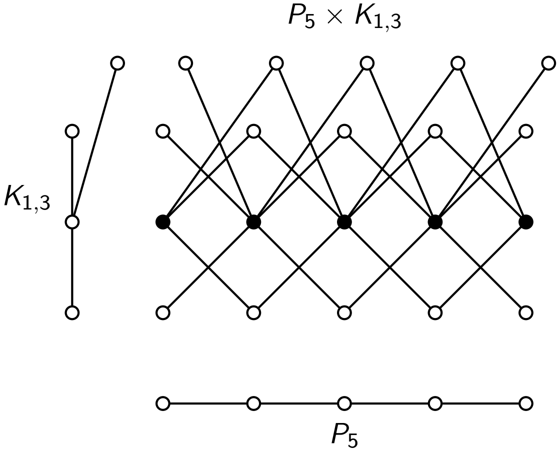

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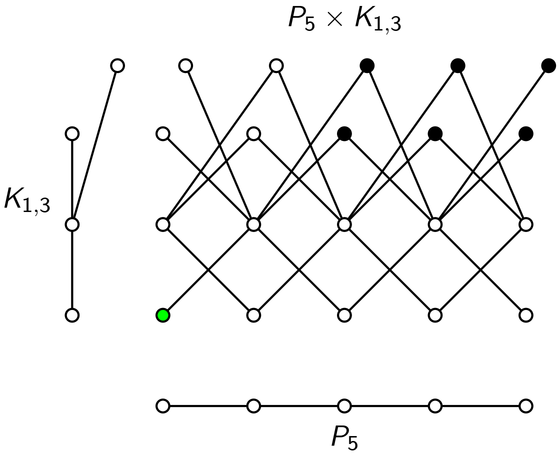

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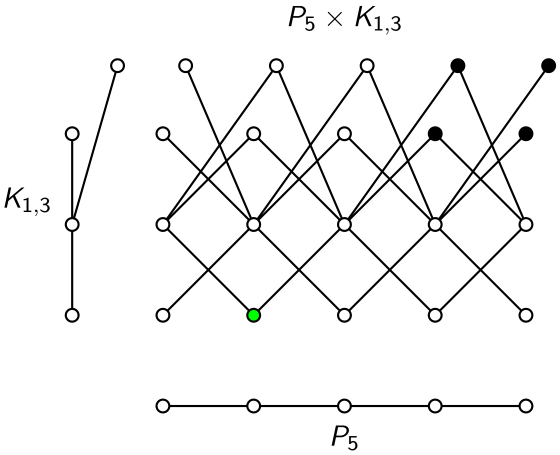

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- Can this decomposition be found by a polynomial algorithm?
- This is well understood for Cartesian, direct and strong product, but not for modular and DcD product.

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- We already observed distance formulas for Cartesian, direct and strong products.
- We continue with some examples for modular (domination number, distance, strong metric dimension) and DcD product (distance).

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- What can we say about $\gamma(G \diamond H)$?

Conjecture (Vizing 1968)

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H)$$

Lower bounds for $\gamma(G \diamond H)$

Proposition

Let G and H be two graphs. If $D = \{(g_1, h_1), \ldots, (g_k, h_k)\}$ is a dominating set in $G \diamond H$, then $\{g_1, \ldots, g_k\}$ is a dominating set in G or $\{h_1, \ldots, h_k\}$ is a total dominating set in $\overline{H}$.

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- If $\mathrm{diam}(H) = 2$, then $\min\{\gamma(G), 3\} \leq \gamma(G \diamond H)$.

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Proposition

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(iv) $\mathrm{diam}(H) \geq 3$ *and there exists a set* $D_G = \{g_1, g_2, g_3\} \subseteq V(G)$ *such that* $\mathrm{pr}[g_3, D_G] = \emptyset$ *and* $N_G[g_1] \cap N_G[g_2] = \emptyset$, *or the mirror condition;*

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- It seems an interesting problem to study $\gamma(\overline{G})$.

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- The above formula holds also when G is not connected and also $G \diamond K_r$ is not connected.

General case

Theorem

If G and H are not complete graphs, then either

- $d_{G\diamond H}((g,h),(g',h')) = \infty$ *when G and H both contain two complete components and (either* $g' \in N_G[g]$ *or* $h' \in N_H[h]$*) or*
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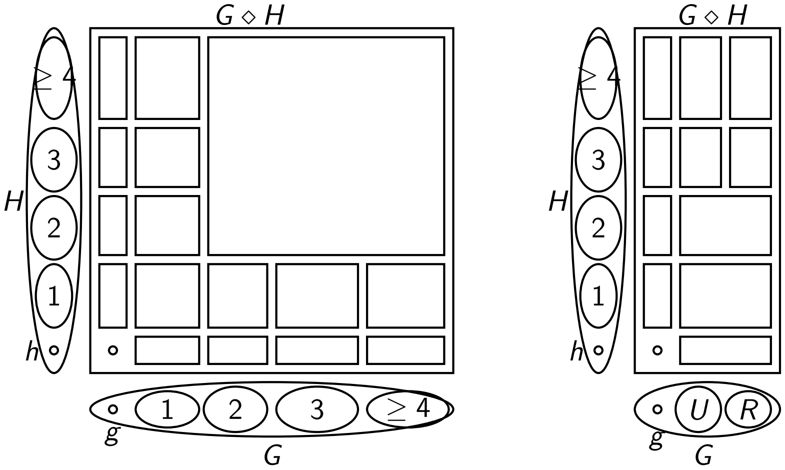
- For the second part we need to find a path of length at most three.

Second part of the proof



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$$N_G[g] = N_G[g'] \wedge d_H(h,h') \geq 3 \wedge \tag{1}$$
$$\big(N_G[g] = V(G) \vee \big(N_G[g] \neq V(G) \wedge \{h,h'\} \text{ is } \gamma(H) - set\big)\big)$$

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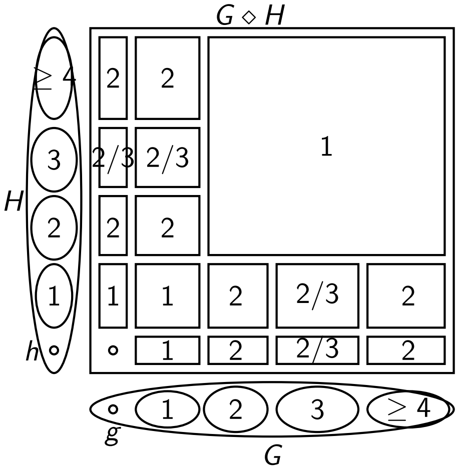
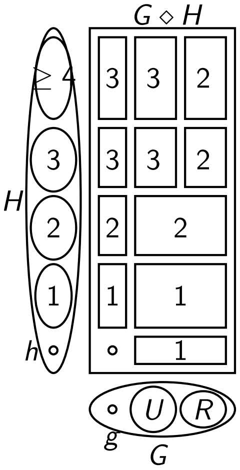
Distance formula

$$d_{G\diamond H}((g,h),(g',h')) = \begin{cases} 0 & : \quad g = g' \wedge h = h' \\ 1 & : \quad (g,h)(g',h') \in E(G \diamond H) \\ 2 & : \quad \text{otherwise} \\ 3 & : \quad (g,h),(g',h') \text{ fulfills (1) or (2)} \end{cases}$$

Schematic overview





Motivation

Corollary

Let graphs G and H be graphs different from the complete graphs and at least one is different than $K_s \cup K_t$. We have $\mathrm{diam}(\mathrm{G} \diamond \mathrm{H}) = 2$ *if and only if (no factor contains a universal vertex and no factor is an efficient closed domination graph with domination number two) or both factors have diameter two.*

Questions and open problems

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Theorem

Let G and H be graphs of order n. The problem of finding clique of order n in $G \circledast H$ is equivalent to isomorphism problem. Moreover, the problem of determining whether $G \circledast H$ has a clique of order $n(1-\epsilon)$ is NP complete.

Non-associativity

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- $(g, h)(g', h) \notin E(G \circledast H)$ and with this $((g, h), f)((g', h), f') \in E((G \circledast H) \circledast F)$.
- $(h, f)(h, f') \notin E(G \circledast H)$ and with this $(g, (h, f))(g', (h, f')) \notin E(G \circledast (H \circledast F))$.

Distance in direct-co-direct product for K_n

- $K_n \circledast H \cong K_n \times H$.

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- For $n \geq 3$ we similarly get $d_{K_n \circledast H}((g,h),(g',h')) =$

$$\begin{cases} \min\{d_H^o(h,h'), \max\{2, d_H^e(h,h')\}\} & : \quad g \neq g' \\ \min\{d_H^e(h,h'), \max\{3, d_H^o(h,h')\}\} & : \quad g = g' \end{cases} .$$

Eccentricity approach

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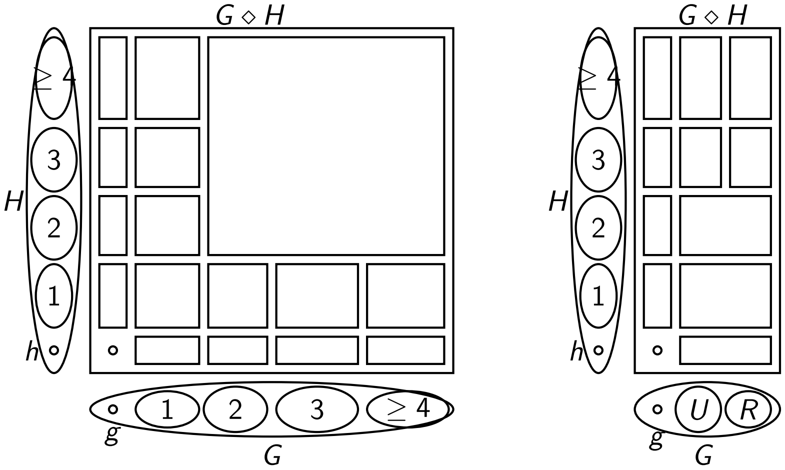
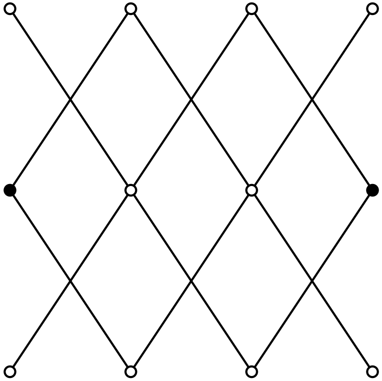
Theorem

Let G and H be two connected graphs at most one isomorphic to $K_{1,t}$ and let $g \in V(G)$ and $h \in V(H)$ such that they are different from a central vertex of a star if G or H, respectively, is isomorphic to a star. If $\mathrm{ecc}_G(g) \geq 3$ or $\mathrm{ecc}_H(h) \geq 3$, then $\mathrm{ecc}_{G \circledast H}((g, h)) \leq 3$.

Sketch of a proof



Problem with a star 1



Problem with a star 2

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Problem with a star 2

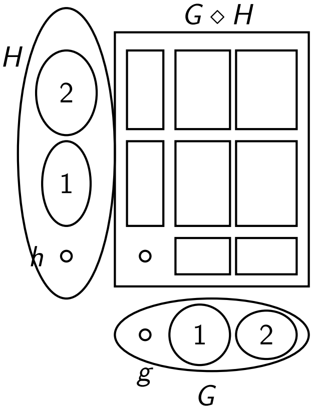
- Let $H = K_{s,t} - e$ where $e = hh'$.
- If g is a universal vertex of $K_{1,t}$, then $d_{K_{1,t} \circledast H}((g,h),(g,h')) = 5$.

Eccentricity 2

Theorem

Let G and H be two connected graphs at least one different from $K_{1,t}$ and let $g \in V(G)$ and $h \in V(H)$. If $\mathrm{ecc}_G(g) = 2$ and $\mathrm{ecc}_H(h) = 2$ and at least one of g and h belongs to C_3, then $\mathrm{ecc}_{G \circledast H}((g, h)) \leq 3$.

Sketch of a proof



Distance 2

Theorem

Let graphs G and H be graphs different from the complete graphs and empty graphs (minus disconected case). The distance $d_{G \circledast H}((g,h),(g',h')) = 2$ *if and only if at least one of the following possibilities holds*

$$(d_H(h,h') = 2 \wedge g = g' \wedge N_G(g) \neq \emptyset) \vee (d_G(g,g') = 2 \wedge h = h' \wedge N_H(h) \neq \emptyset)$$

$$(d_{\overline{H}}(h,h') = 2 \wedge g = g' \wedge N_{\overline{G}}(g) \neq \emptyset) \vee (d_{\overline{G}}(g,g') = 2 \wedge h = h' \wedge N_{\overline{H}}(h) \neq \emptyset)$$

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$$d_{G \circledast H}((g,h),(g',h')) = \begin{cases} 0 & : \quad g = g' \wedge h = h' \\ 1 & : \quad (g,h)(g',h') \in E(G \diamond H) \\ 2 & : \quad (g,h),(g',h') \text{ fulfills previous theorem} \\ 3 & : \quad \text{otherwise} \\ 4 & : \quad \text{condition (A)} \\ 5 & : \quad \text{condition (B)} \end{cases}$$

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- Condition (A) consists of 10 different conditions where one factor is always $K_{1,t}$
- For Condition (B) let $G \cong K_{1,t}$, $t \geq 2$ where $g = g'$ is universal in G and $\mathrm{ecc}_H(h) = 3$, h' is an isolated vertex of $H[C_H]$, $H[A_H]$ and $H[N_H(h')]$ are without edges, every vertex of A_H is adjacent to every vertex of B_H and $C_H \subset N_H(h_0)$ for every $h_0 \in N_H(h')$ (or symmetric).

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Direct-co-direct product is not connected if and only if

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Strong metric dimension

- A vertex $z \in V(G)$ *strongly resolves* two different vertices $x, y \in V(G)$ if x belongs to a y, z-geodesic, or y belongs to a x, z-geodesic.

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- A *strong metric generator* in a connected graph G is a set $S \subseteq V(G)$ such that every two vertices of G are strongly resolved by a vertex of S.
- By $\dim_s(G)$ we denote the smallest cardinality of a strong metric generator for G and we call it the *strong metric dimension* of G.

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- Moreover, $uv \in E(G_{SR})$ if there exists a maximal u, v-geodesic in G.

Theorem (Oellermann and Peters-Fransen)

For any connected graph G, $\dim_s(G) = \beta(G_{SR})$.

Strong resolving graph of $G \diamond H$

Theorem

For non-complete graphs G and H where at least one is different than $K_m \cup K_n$ we have $(g, h)(g', h') \in E((G \diamond H)_{SR})$ if and only if

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- *$d_{G \diamond H}((g,h),(g',h')) = 3$,*

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- $d_{G \diamond H}((g,h),(g',h')) = 3$*, or*
- (g,h) *or* (g',h') *is a 3-diametrical vertex where* $\{x,x'\}$ *are universal in* X *and* $d_Y(y,y') = 2$ *and* $yy' \in E(Y_{SR})$ *for* $\{X,Y\} = \{G,H\}$ *and* $\{x,x',y,y'\} = \{g,g',h,h'\}$*,*

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- (g,h) *is a 3-diametrical vertex where* x *is universal and* x' *is not universal in* X *and* $d_Y(y,y') = 2$ *and* $d_Y(y,y_0) \leq 2$ *for every* $y_0 \in N_Y(y')$ *for* $\{X,Y\} = \{G,H\}$ *and* $\{x,x',y,y'\} = \{g,g',h,h'\}$.

$\mathrm{diam}(G \diamond H) \leq 2$

Theorem

Let G and H be graphs. If $\mathrm{diam}(G \diamond H) \leq 2$, then $E((G \diamond H)_{SR})$ equals to

$$TW(G \diamond H) \cup E(\overline{G} \square \overline{H}) \cup E(G \times \overline{H}) \cup E(\overline{G} \times H).$$

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Proposition

For integers $s \geq t \geq 2$ we have $\dim_s(K_{1,s} \diamond K_{1,t}) = st + s - 1$.

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Proposition

For integers $s, t \geq 5$, $\max\{s, t\} \geq 6$, we have $\dim_s(\overline{C}_s \diamond \overline{C}_t) = st - \lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor$. In addition, $\dim_s(\overline{C}_5 \diamond \overline{C}_5) = 20$.

Another familly

Proposition

For integers $s, t \geq 7$ we have
$\dim_s(C_s \diamond C_t) = st - 4\min\{\left\lfloor \frac{s}{3} \right\rfloor, \left\lfloor \frac{t}{3} \right\rfloor\} - r$*, where*

$$r = \begin{cases} 0 & : \quad \min\{s,t\} \equiv \{0,1\} \pmod 3 \\ 1 & : \quad s = t \wedge \min\{s,t\} \equiv 2 \pmod 3 \\ 2 & : \quad s \neq t \wedge \min\{s,t\} \equiv 2 \pmod 3 \end{cases} .$$

One factor has a γ-pair and the other has no universal vertex

Theorem

If a graph G has a γ_G-pair and a graph H is without a universal vertex, then $E((G \diamond H)_{SR})$ equals to

$$TW(G \diamond H) \cup E(GP(G) \square GP(H)) \cup E(\overline{G}^{-} \square \overline{H}^{-}) \cup E(G^{-} \times \overline{H}^{-}) \cup E(\overline{G}^{-} \times H^{-})$$

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For integers $n, p \geq 3$ we have $\dim_s(K_{n,n}^{-M} \diamond K_{p,p}^{-M}) = 3np$.

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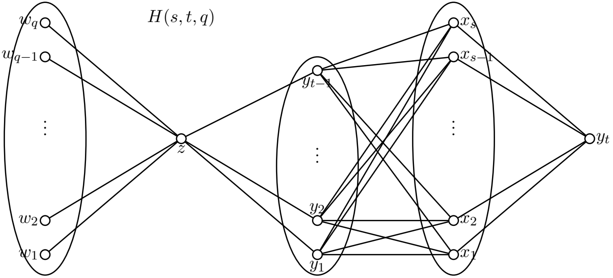
For integer $r \geq 7$ we have $\dim_s(P_5 \diamond P_r) = \dim_s(P_5 \diamond C_r) = 3r - 2$.

One factor has a universal vertex and the other is arbitrary



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Proposition

For integers $r, q \geq 3$*,* $s, t \geq 4$*,* $m_q = \min\{r, q\}$*,* $m_t = \min\{t-1, r-q\}$*,* $m_s = \min\{s, r-q\}$ *and*

$$b = \begin{cases} r+2 & : \quad r \leq q+1 \\ r+1 & : \quad r = q+2 \vee (r \geq q+3 \wedge \max\{s+1, t\} \geq r) \\ q+m_s & : \quad r \geq q+3 \wedge t \leq s < r-1 \\ q+m_t & : \quad r \geq q+3 \wedge s < t < r \end{cases},$$

we have

$$\dim_s(K_{1,r} \diamond H(s,t,q)) = (s+t+q-1)r - b + r + q + s + t.$$

Thank you for your attention!