Grundy domination invariants

Tanja Dravec

University of Maribor, Faculty of Natural Sciences and Mathematics, Slovenia

Institute of Mathematics, Physics and Mechanics, Ljubljana

Indo-Slovenia Pre-Conference School on Algorithms and Combinatorics, February 2024

1. Definitions and examples
2. Relations between Grundy domination invariants
3. Bounds for Grundy domination numbers
4. Complexity results
5. Open problems

Domination

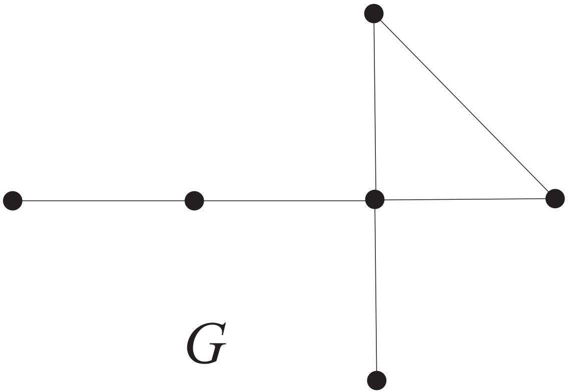
Definition

Let G be a graph. A set $D \subseteq V(G)$ is a **dominating set** of G if

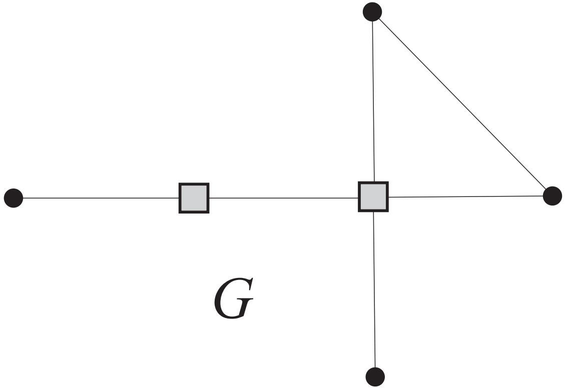
$$\bigcup_{u \in D} N[u] = V(G).$$

The cardinality of a minimum dominating set of a graph G is called the **domination number** of G, denoted $\gamma(G)$.

Example



Example



Total domination

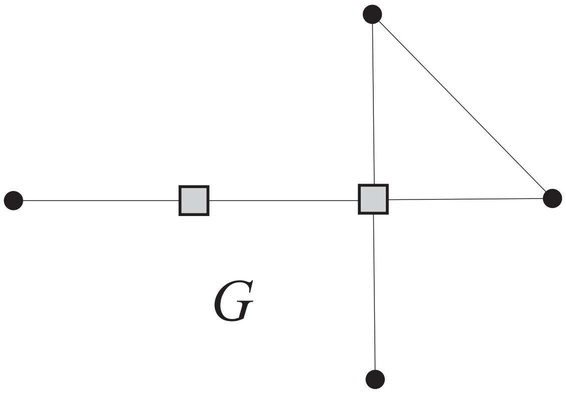
Definition

Let G be a graph. A set $D \subseteq V(G)$ is a **total dominating set** of G if

$$\bigcup_{u \in D} N(u) = V(G).$$

The cardinality of a minimum total dominating set of a graph G is called the **total domination number** of G, denoted $\gamma_t(G)$.

Example

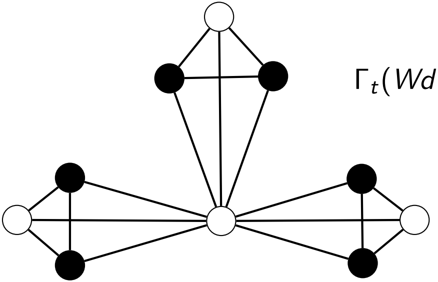


Upper total domination

Definition

The maximum cardinality of a minimal total dominating set of a graph G is called the **upper total domination number** of G, denoted $\Gamma_t(G)$.

Windmill graph Wd(k, n) is obtained by taking n vertex disjoint copies of the complete graph K_k, selecting one vertex from each copy, and identifying these n selected vertices into one new vertex.



$$\Gamma_t(Wd(4,3)) = 6$$

Grundy domination number

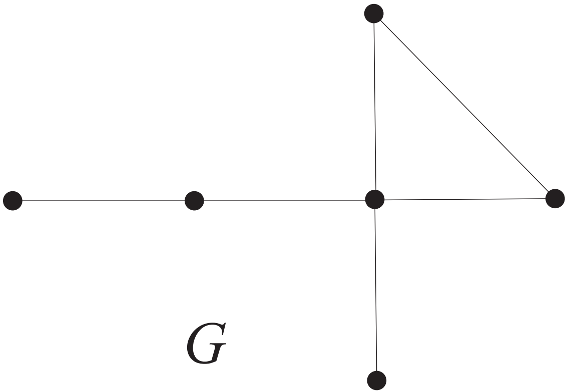
- Let $S = (v_1, \ldots, v_k)$ be a sequence of vertices of a graph G. The corresponding set of vertices from S will be denoted by by $\hat{S}$.
- A sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of a graph G is called a closed neighborhood sequence if, for each $i \in \{2, \ldots, k\}$

$$N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j] \neq \emptyset.$$

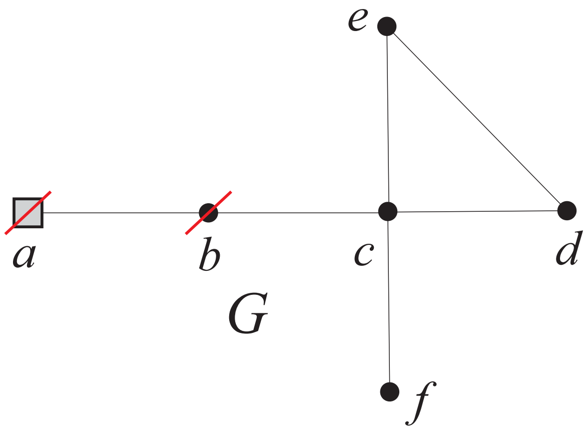
- The maximum length of a closed neighborhood sequence in a graph G is the **Grundy domination number** of G, denoted by $\gamma_{gr}(G)$. The corresponding sequence is called a *Grundy dominating sequence* of a graph.
- For any graph G, $\gamma_{gr}(G) \geq \gamma(G)$.

[B,2014] B. Brešar, T. G., M. Milanič, D. F. Rall, R. Rizzi, Dominating sequences in graphs, *Discrete Math.* 336 (2014) 22–36.

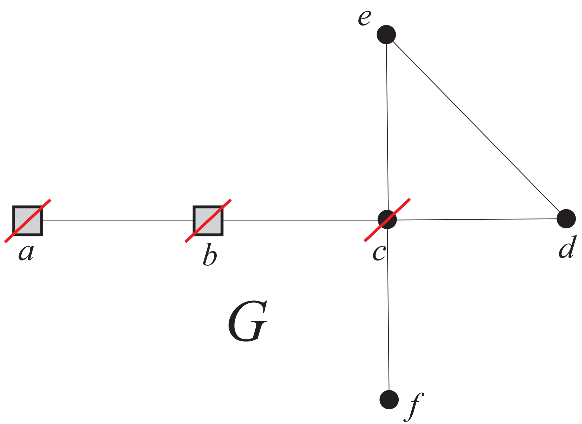
Grundy dominating sequence



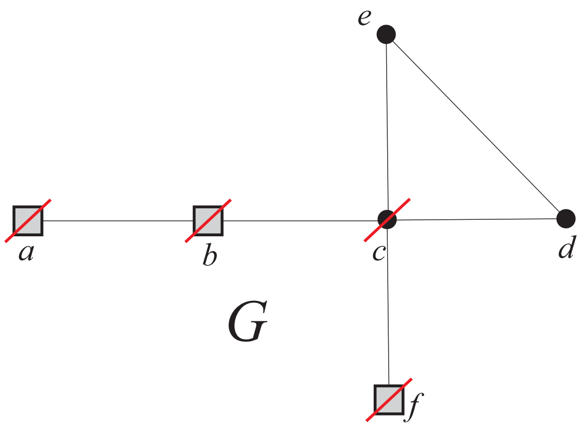
Grundy dominating sequence



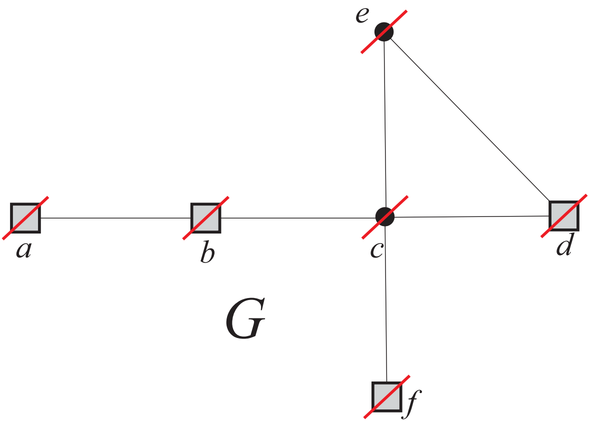
Grundy dominating sequence



Grundy dominating sequence



Grundy dominating sequence



Footprinter

- Let $S = (v_1, \ldots, v_k)$ be a closed neighborhood sequence. We say that vertex v_i *footprints* the vertices from $N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$, and that v_i is their **footprinter**.
- Let $f_S \colon V(G) \to \widehat{S}$ be a function that maps each vertex to its footprinter.

Grundy total domination number

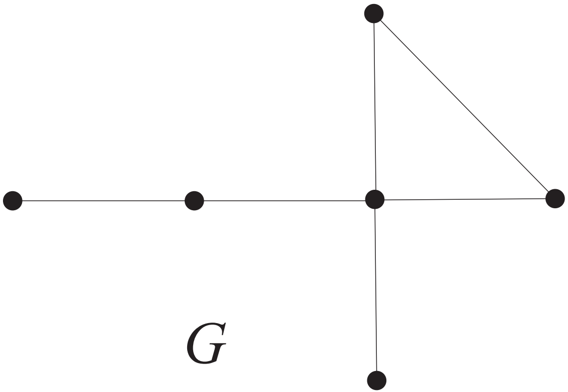
- A sequence $S = (v_1, \ldots, v_k)$ of vertices of a graph G is an open neighborhood sequence, if for every $i \in \{2, \ldots, k\}$

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset.$$

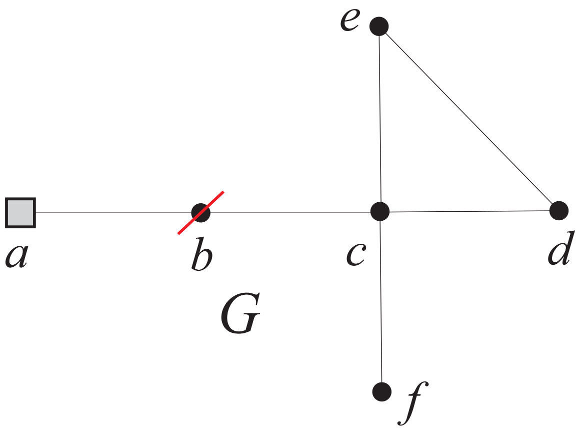
- The maximum length of an open neighborhood sequence in G is the *Grundy total domination number* of G and is denoted by $\gamma_{\rm gr}^t(G)$.
- The corresponding sequence is called a *Grundy total dominating sequence* of a graph.
- If G is a graph without isolated vertices, then $\gamma_{\rm gr}^t(G) \geq \gamma_t(G)$.

[BHR,2016] B. Brešar, M. A. Henning, D. F. Rall, Total dominating sequences in graphs, *Discrete Math.* 339 (2016) 1665–1676.

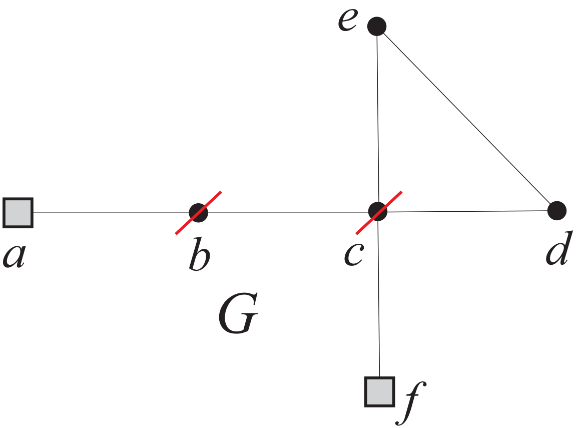
Grundy total domination - example



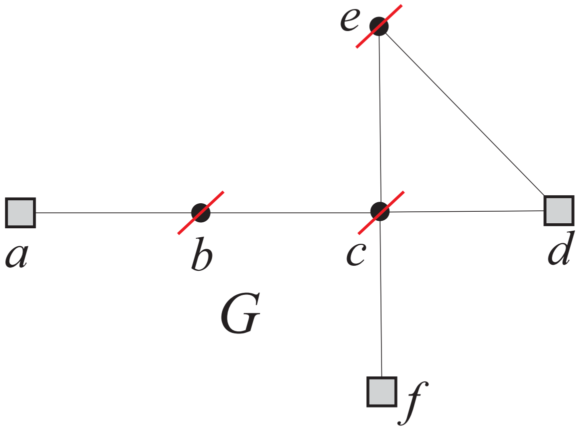
Grundy total domination - example



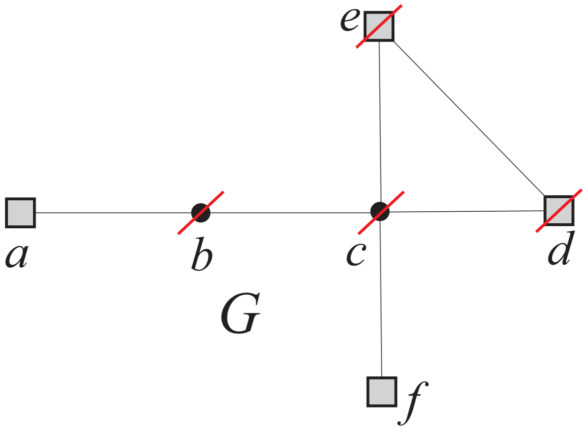
Grundy total domination - example



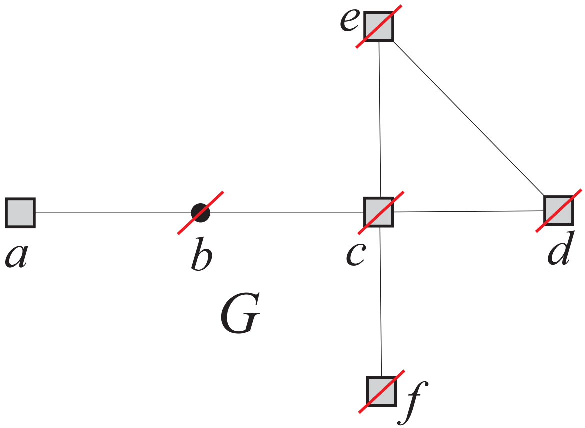
Grundy total domination - example



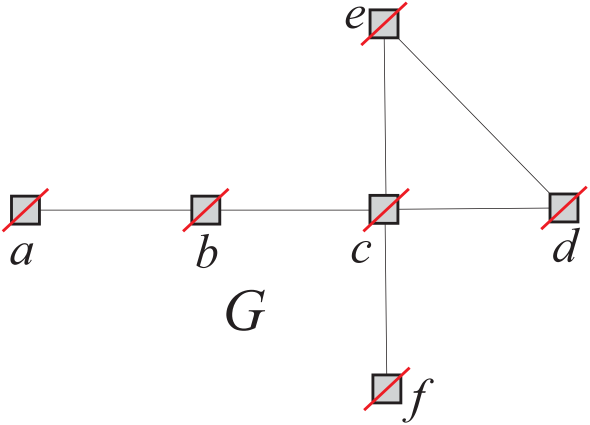
Grundy total domination - example



Grundy total domination - example



Grundy total domination - example



Two more invariants

- A sequence $S = (v_1, \ldots, v_k)$ of vertices of a graph G is a Z-sequence, if for every $i \in \{2, \ldots, k\}$

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset.$$

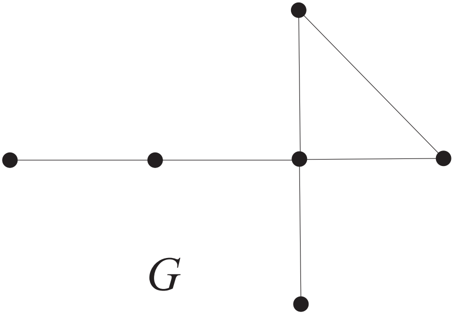
- The maximum length of a Z-sequence in G is the *Z-Grundy domination number* of G and is denoted by $\gamma_{\rm gr}^{Z}(G)$.
- A sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of a graph G is an L-sequence, if for every $i \in \{2, \ldots, k\}$

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset.$$

- The maximum length of an L-sequence in G is the *L-Grundy domination number* of G and is denoted by $\gamma_{\rm gr}^{L}(G)$.

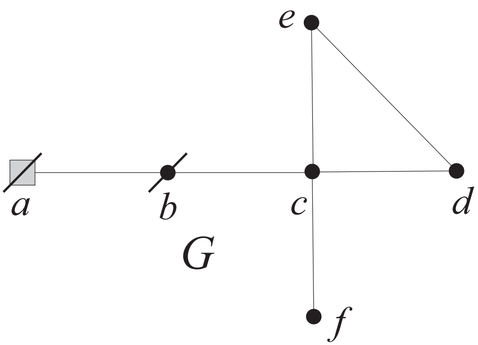
Z-Grundy domination - example

[B,2017] B. Brešar, C. Bujtás, T. G., S. Klavžar, G. Košmrlj, B. Patkós, Z. Tuza, M. Vizer, Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77.



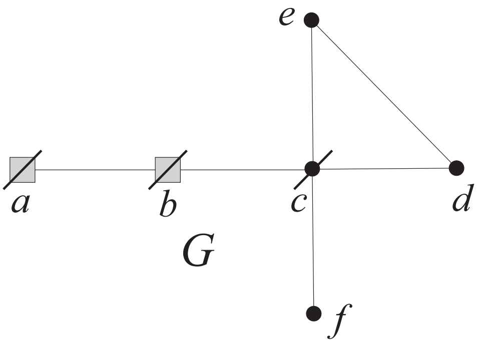
Z-Grundy domination - example

[B,2017] B. Brešar, C. Bujtás, T. G., S. Klavžar, G. Košmrlj, B. Patkós, Z. Tuza, M. Vizer, Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77.



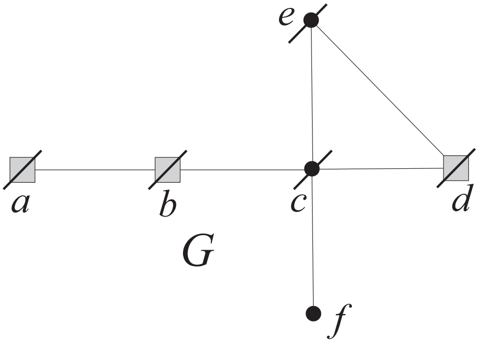
Z-Grundy domination - example

[B,2017] B. Brešar, C. Bujtás, T. G., S. Klavžar, G. Košmrlj, B. Patkós, Z. Tuza, M. Vizer, Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77.



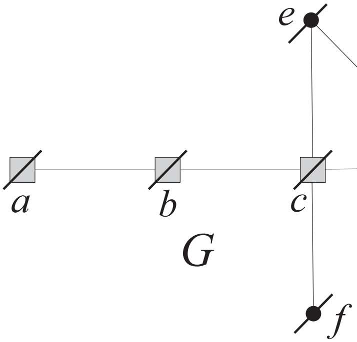
Z-Grundy domination - example

[B,2017] B. Brešar, C. Bujtás, T. G., S. Klavžar, G. Košmrlj, B. Patkós, Z. Tuza, M. Vizer, Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77.



Z-Grundy domination - example

[B,2017] B. Brešar, C. Bujtás, T. G., S. Klavžar, G. Košmrlj, B. Patkós, Z. Tuza, M. Vizer, Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77.



Zero forcing sets

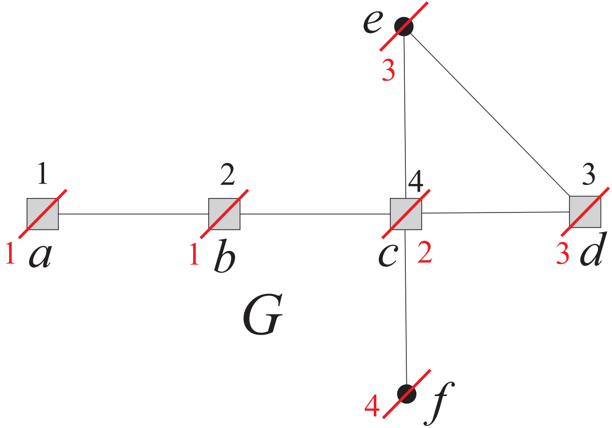
- Color vertices of a graph G white and blue.
- *Color change rule:* If a given blue vertex has exactly one white neighbor w, then the color of w is changed to blue.
- A *zero forcing set* for G is a subset B of its vertices such that if initially vertices from B are colored blue and the remaining vertices are colored white, then by a repeated application of the color change rule all the vertices of G are turned to blue.
- The **zero forcing number** $Z(G)$ of a graph G is the size of a minimum zero forcing set.

[AIM,2008] AIM Minimum Rank-Special Graphs Work Group, Zero-forcing sets and the minimum rank of graphs, Linear Algebra Appl. 428 (2008) 1628–1648.

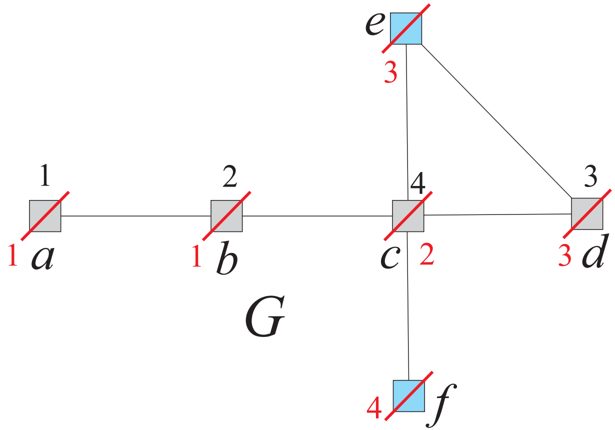
Minimum rank and maximum nullity

- Let G be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$. The **minimum rank** $\mathrm{mr}(G)$ of G is the smallest possible rank over all symmetric real matrices whose (i, j)-th entry, $i \neq j$, is nonzero whenever vertices i and j are adjacent in G and is zero otherwise. (There are no restrictions on the diagonal entries.)
- The **maximum nullity** $\mathrm{M}(G)$ of G is the biggest possible nullity over all the above matrices.
- $\mathrm{M}(G)+\mathrm{mr}(G)= |V(G)|$.
- **Thm.** [AIM,2008] For any graph G, $|V(G)|-\mathrm{mr}(G)=\mathrm{M}(G) \leq Z(G)$.

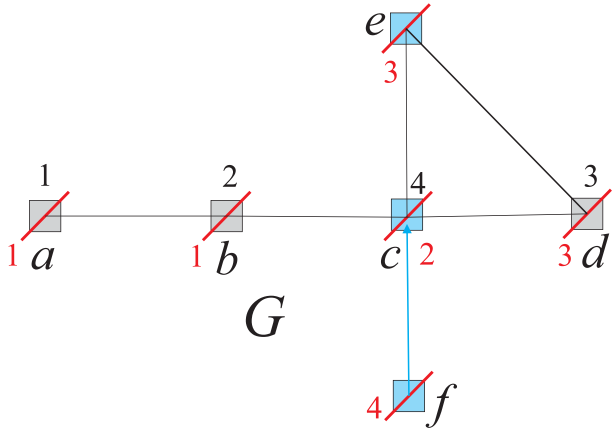
Z-Grundy domination vs. zero forcing



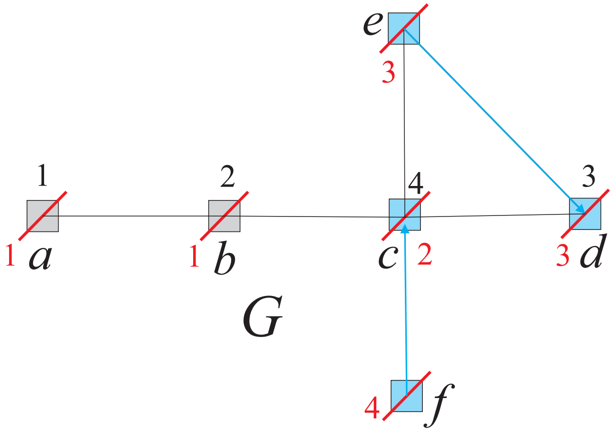
Z-Grundy domination vs. zero forcing



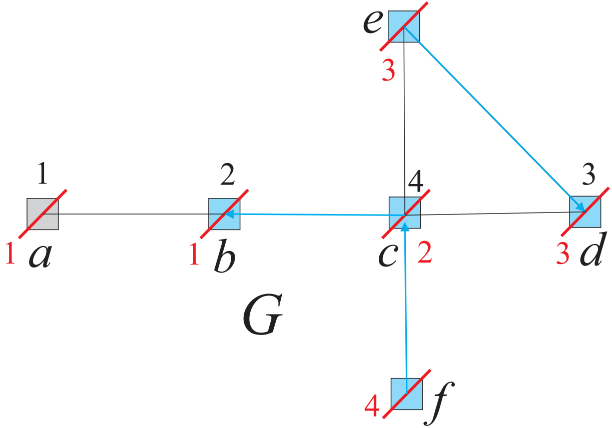
Z-Grundy domination vs. zero forcing



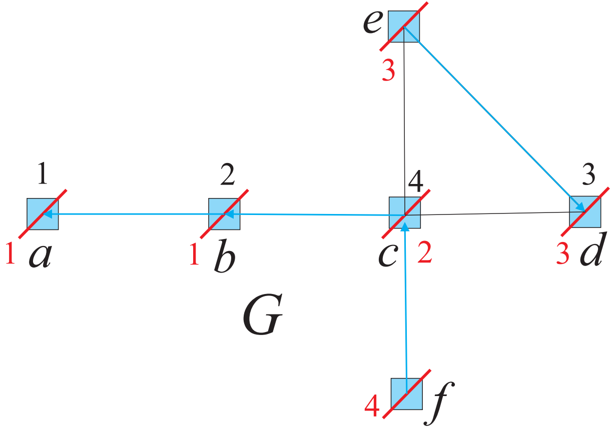
Z-Grundy domination vs. zero forcing



Z-Grundy domination vs. zero forcing



Z-Grundy domination vs. zero forcing



Connection of $\gamma_{\rm gr}^Z(G)$ and $Z(G)$

Theorem (B,2017)

If G is a graph, then

$$\gamma_{\rm gr}^Z(G) + Z(G) = |V(G)|\,.$$

Moreover, the complement of a (minimum) zero forcing set of G is a (maximum) Z-set of G and vice versa.

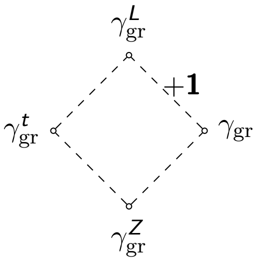
Relations between Grundy domination numbers

Proposition (B, 2017)

If G is a graph with no isolated vertices, then

1. $\gamma_{\rm gr}^{Z}(G) \leq \gamma_{\rm gr}(G) \leq \gamma_{\rm gr}^{L}(G) - 1$,
2. $\gamma_{\rm gr}^{Z}(G) \leq \gamma_{\rm gr}^{t}(G) \leq \gamma_{\rm gr}^{L}(G)$,

and all the bounds are sharp.



Relations between Grundy domination numbers

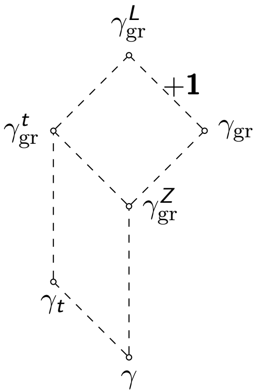
Theorem (B,2017)

If G is a graph (without isolated vertices), then

- $\gamma_{\rm gr}^{t}(G) \leq 2\gamma_{\rm gr}^{Z}(G)$,
- $\gamma_{\rm gr}^{L}(G) \leq 2\gamma_{\rm gr}(G)$.

- $\gamma_{\rm gr}^{t}(Wd(k,n)) = 2n = 2\gamma_{\rm gr}^{Z}(Wd(k,n))$,
- $\gamma_{\rm gr}^{L}(Wd(k,n)) = 2n = 2\gamma_{\rm gr}(Wd(k,n))$.

Relations between domination invariants



Relations between Grundy domination parameters and domination parameters

Theorem (BCDH,2023+)

If G is a graph with no clique component, then $\gamma_t(G) \leq \gamma_{\mathrm{gr}}^{Z}(G)$ or equivalently $Z(G) \leq |V(G)| - \gamma_t(G)$.

Proof. For a total dominating set D of a graph G we use the following notations:

- let $C_1, \ldots, C_\ell$ be the K_2 components of $G[D]$, $V(C_i) = \{x_i, y_i\}$,
- for each $i \in [\ell]$, let $A_i(D)$ denote the set of vertices that are totally dominated by $V(C_i)$ and are not totally dominated by $D \setminus V(C_i)$,
- For $k \leq \ell$ denote by $C_1, \ldots, C_k$ K_2-components of $G[D]$ with the property $N[x_i] \cap A_i(D) = N[y_i] \cup A_i(D)$ and by $C_{k+1}, \ldots, C_\ell$ the components without this property.

Proof

Let D be a γ_t-set of G such that $G[D]$ has the smallest possible number of K_2-components (among all γ_t-sets of G).

Proof

Let D be a γ_t-set of G such that $G[D]$ has the smallest possible number of K_2-components (among all γ_t-sets of G).

For any component C of $G[D]$, not isomorpic to K_2, we do the following:

- first add to S all vertices u in $G[D]$ such that u is adjacent to a vertex $v \in V(C)$ with $\deg_C(v) = 1$;

Proof

Let D be a γ_t-set of G such that $G[D]$ has the smallest possible number of K_2-components (among all γ_t-sets of G).

For any component C of $G[D]$, not isomorpic to K_2, we do the following:

- first add to S all vertices u in $G[D]$ such that u is adjacent to a vertex $v \in V(C)$ with $\deg_C(v) = 1$;
- add to S all the remaining vertices in C.

Proof

Let D be a γ_t-set of G such that $G[D]$ has the smallest possible number of K_2-components (among all γ_t-sets of G).

For any component C of $G[D]$, not isomorpic to K_2, we do the following:

- first add to S all vertices u in $G[D]$ such that u is adjacent to a vertex $v \in V(C)$ with $\deg_C(v) = 1$;
- add to S all the remaining vertices in C.

For any K_2 component $C_i \in \{C_{k+1}, \ldots, C_\ell\}$ (K_2 component with the property $N[x_i] \cap A_i \neq N[y_i] \cap A_i$) do the following (we may WLOG assuem that $\big(N(x_i) \setminus N[y_i]\big) \cap A_i \neq \emptyset$):

Proof

- let $a_i \in \big(N(x_i) \setminus N[y_i]\big) \cap A_i$;

Proof

- let $a_i \in \big(N(x_i) \setminus N[y_i]\big) \cap A_i$;
- we first add to S the vertex y_i, and then the vertex x_i.

Proof

- let $a_i \in \big(N(x_i) \setminus N[y_i]\big) \cap A_i$;
- we first add to S the vertex y_i, and then the vertex x_i.

For each K_2 component $C_i \in \{C_1, \ldots, C_k\}$ (with the property $N[x_i] \cap A_i = N[y_i] \cap A_i$) do the following:

Proof

- let $a_i \in \big(N(x_i) \setminus N[y_i]\big) \cap A_i$;
- we first add to S the vertex y_i, and then the vertex x_i.

For each K_2 component $C_i \in \{C_1, \ldots, C_k\}$ (with the property $N[x_i] \cap A_i = N[y_i] \cap A_i$) do the following:

- A_i does not induces a complete graph.

Proof

- let $a_i \in \big(N(x_i) \setminus N[y_i]\big) \cap A_i$;
- we first add to S the vertex y_i, and then the vertex x_i.

For each K_2 component $C_i \in \{C_1, \ldots, C_k\}$ (with the property $N[x_i] \cap A_i = N[y_i] \cap A_i$) do the following:

- A_i does not induces a complete graph.
- For each $i \in [k]$, there exists a vertex $a_i \in A_i \setminus \{x_i, y_i\}$ such that a_i is not adjacent to a vertex $b_i \in A_i \setminus \{x_i, y_i\}$.

Proof

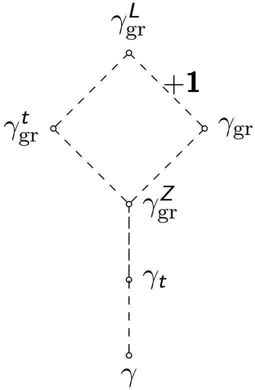
- let $a_i \in \big(N(x_i) \setminus N[y_i]\big) \cap A_i$;
- we first add to S the vertex y_i, and then the vertex x_i.

For each K_2 component $C_i \in \{C_1, \ldots, C_k\}$ (with the property $N[x_i] \cap A_i = N[y_i] \cap A_i$) do the following:

- A_i does not induces a complete graph.
- For each $i \in [k]$, there exists a vertex $a_i \in A_i \setminus \{x_i, y_i\}$ such that a_i is not adjacent to a vertex $b_i \in A_i \setminus \{x_i, y_i\}$.
- For each $i \in [k]$ and $x \in V(C_i)$, x has no neighbors in $V(G) - A_i(D)$.
- For any $i \neq j \in \{1, \ldots, k\}$ there are no edges between $A_i(D)$ and $A_j(D)$.
- For any $i \in \{1, \ldots, k\}$ add to S a_i (which footprints x_i) and then x_i (which footprints b_i).

Relations between domination invariants

If G is a a graph with no clique component, then:



Relations between Grundy domination parameters and domination parameters

Theorem (BCDH,2023+)

If G is a graph with no clique component, then $\gamma_t(G) \leq \gamma_{\rm gr}^{Z}(G)$ or equivalently $Z(G) \leq |V(G)| - \gamma_t(G)$.

- The bound is sharp: $\gamma_{\rm gr}^{Z}(K_{1,\ell}) = 2 = \gamma_t(K_{1,\ell})$;
- The ration $\frac{\gamma_{\rm gr}^{Z}(G)}{\gamma_t(G)}$ can be arbitrary large.

[BCDH,2023+] B. Brešar, M.G. Cornet, T. D., M. Henning, Bounds on zero forcing using (upper) total domination and minimum degree, submitted.

Uniform graphs for total dominating sequences

A graph G is total k-uniform if $\gamma_t(G) = \gamma_{gr}^t(G) = k$.

Proposition (BHR,2016)

A graph G is total 2-uniform if and only if G is complete multipartite graph.

Theorem (BK,2018)

There exists no graph G such that $\gamma_{\mathrm{gr}}^t(G) = 3$.
For any $n \in \mathbb{Z}^+ \setminus \{1, 3\}$ exists a graph G_n with $\gamma_{\mathrm{gr}}^t(G_n) = n$.

[BK,2018] B. Brešar, T. Kos, G. Nasini, P. Torres, Total dominating sequences in trees, split graphs, and under modular decomposition, Discrete Optim. 28 (2018) 16–30.

Total k-uniform graphs

[GJ,2021] T. G., M. Jakovac, T. Kos, T. Marc, On graphs with equal total domination and Grundy total domination number, Aequationes Math. (2021) 1–10.
[BGD,2021] S. Bahadır, D. Gözüpek, O. Doğan, On graphs all of whose total dominating sequences have the same length, Discrete Math. 344 (2021) 112492

Corollary (GJ,2021)

Let G be a bipartite graph with bipartition $A \cup B$. Then the Grundy total domination number of G is even and for any Grundy total dominating sequence $S = (v_1, \ldots, v_{2k})$ it follows that $|A \cap \hat{S}| = |B \cap \hat{S}| = k$.

For odd k there is no total k-uniform bipartite graph G.

Total k-uniform graphs

Lemma (BGD,2021)

Let G be a total k-uniform graph with no isolated vertices where $k \geq 3$. If $uv \in E(G)$, then $G - (N[u] \cup N[v])$ is a total $(k-2)$-uniform graph with no isolated vertex.

Theorem (BGD,2021)

There does not exist a total k-uniform graph where k is an odd positive integer.

Total 4-uniform graphs

Theorem (GJ,2021)

A false twin-free bipartite graph G is total 4-uniform if and only if G is isomorphic to $K_{n,n} - M$, $n \geq 2$, where M is a perfect matching of $K_{n,n}$.

Conjecture (GJ,2021)

Let G be a connected false twin-free graph. Then $\gamma_t(G) = \gamma_{\rm gr}^t(G) = 4$ if and only if G is isomorphic to the graph $K_{n,n} - M$, $n \geq 3$, where M denotes an arbitrary perfect matching of $K_{n,n}$.

Theorem (GJ,2021)

There is no connected chordal total 4-uniform graph.

Total 4-uniform graphs

The conjecture was disproved.

Proposition (BGD,2021)

The graphs $L(K_6)$ is total 4-uniform and non bipartite.

Theorem (BGD,2021)

There is no connected chordal total k-uniform graph for $k \geq 4$.

Total k-uniform graphs

Theorem (BGD,2021)

If the graph G is connected, non-bipartite and total k-uniform, then the direct product $G \times K_2$ is a connected total $2k$-uniform graph.

Theorem (BGD,2021)

For every positive even integer k, a connected, false twin-free and total k-uniform graph is regular.

Problem

For even $k \geq 4$, characterize all connected total k-uniform graphs.

Total domination vs Z-Grundy domination

Theorem (BCDH,2023+)

Let G be a connected graph not isomorphic to a complete graph. Then $\gamma_t(G) = \gamma_{\mathrm{gr}}^Z(G) = 2$ if and only if $N[x] \cup N[y] = V(G)$ holds for any non-twin vertices $x, y \in V(G)$.

Theorem (BCDH,2023+)

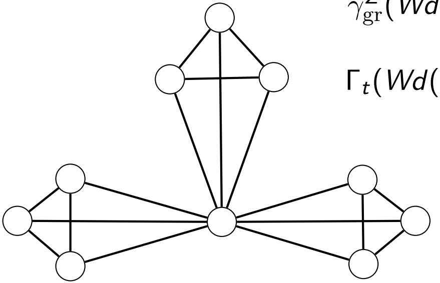
If G is a connected graph that contains a simplicial vertex, then $\gamma_t(G) \neq 3$ or $\gamma_{\mathrm{gr}}^Z(G) \neq 3$.

Problem (BCDH,2023+)

Is there a connected chordal graph G with $\gamma_t(G) = \gamma_{\mathrm{gr}}^Z(G) = k$ for $k \geq 4$?

Upper total domination vs Z-Grundy domination

- There exist graphs such that the upper total domination number is larger than the Z-Grundy domination number, and the difference can even be arbitrary large (windmill graphs WD(k, n), where $k \geq 3, n \geq 2$).



$\gamma_{\mathrm{gr}}^{Z}(Wd(k,n)) = n$

$\Gamma_t(Wd(k,n)) = 2n$

Upper total domination vs Z-Grundy domination

Theorem (BCDH,2023+)

If G is an isolate-free graph, then $\Gamma_t(G) \leq 2\gamma_{\rm gr}^Z(G)$.

Proof.

- Let D be a Γ_t-set of G, $S = ()$.
- For any component C of $G[D]$ not isomorpic to K_2, first add to S all vertices of C that are neighbors of leaves of C, then add to S all remaining vertices of S.
- For each K_2 component C of $G[D]$ add to S one vertex from C.
- D is a Z-sequence of cardinality at least $\frac{\Gamma_t(G)}{2}$.

Problem

Characterize graphs G with $\Gamma_t(G) = 2\gamma_{\rm gr}^Z(G)$.

Graphs with $\Gamma_t(G) = 2\gamma_{\rm gr}^Z(G)$

We already obtained some necessary conditions.

Proposition (BCDH,2023+)

If G is a graph with $\Gamma_t(G) = 2\gamma_{\rm gr}^Z(G)$ and D is a Γ_t-set of G, then the following properties hold.

- (i) *each component of $G[D]$ is isomorphic to K_2 and so $|D| = 2\ell$, for some integer ℓ;*
- (ii) $N[x_i] \cap A_i(D) = N[y_i] \cap A_i(D)$, *for each $i \in [\ell]$;*
- (iii) $A_i(D)$ *induces a clique, for each $i \in [\ell]$;*
- (iv) *there are no edges between $A_i(D)$ and $A_j(D)$, for each $\{i,j\} \subset [\ell]$;*
- (v) *vertices in $A_i(D)$ are closed twins, for each $i \in [\ell]$;*

Upper total domination vs Z-Grundy domination

- Every graph H is an induced subgraph of a graph G with $\Gamma_t(G) = 2\gamma_{\rm gr}^Z(G)$.
- In many graphs the Z-Grundy domination number is much bigger than upper total domination number.
- The ratio $\frac{\gamma_{\rm gr}^Z(G)}{\Gamma_t(G)}$ can be arbitrary large.

Trivial bounds

Let G be an arbitrary graph and $\delta(G)$ the minimum degree of G. Then

- $\gamma_{\rm gr}^Z(G) \leq \gamma_{\rm gr}(G) \leq |V(G)| - \delta(G)$; [B,2014, B,2017]
- $\gamma_{\rm gr}^t(G) \leq |V(G)| - \delta(G) + 1$; [BHR,2016]
- $\gamma_{\rm gr}^L(G) \leq |V(G)| - \delta(G) + 1$; [HS,2022]

[HS,2022] R. Herrman, G.Z. Smith, On the length of L-Grundy sequences, Discrete Optim. 45 (2022) 100725.

Open problems

- Characterize graphs G with $\gamma_{\mathrm{gr}}(G) = |V(G)| - \delta(G)$.
- Characterize graphs G with $\gamma_{\mathrm{gr}}^{t}(G) = |V(G)| - \delta(G) + 1$.
- Characterize graphs G with $\gamma_{\mathrm{gr}}^{L}(G) = |V(G)| - \delta(G) + 1$.

For L-Grundy domination number this characterization is not known even for graphs with $\delta(G) = 1$.

NP-completness

- Grundy domination number problem is NP-complete even when restricted to chordal graphs; [B,2014]
- Grundy domination number problem is NP-complete even when restricted to bipartite and co-bipartite graphs; [BPS,2023]
- Grundy total domination number problem is NP-complete even when restricted to bipartite graphs; [BHR,2016]
- Z-Grundy domination number problem is NP-complete; [Aazami,2008]
- L-Grundy domination number problem is NP-complete even when restricted to bipartite graphs; [B,2017]

[BPS,2023] B. Brešar, A. Pandey, G. Sharma, Computation of Grundy dominating sequences in (co-)bipartite graphs, Comput. Appl. Math. 42 (2023) 359.

Aazami,2008 A. Aazami, Hardness results and approximation algorithms for some problems on graphs (Ph.D. thesis),University of Waterloo,2008.

Bipartite graphs

Question: Is there a polynomial time algorithm that computes Z-Grundy domination number of a bipartite graph?

Trees, $\gamma_{\rm gr}, \gamma_{\rm gr}^t$

- There exists a linear time algorithm for the Grundy domination number of a tree. [B,2014]
- No explicit formula for computing Grundy domination number of a tree exists.
- There exists a linear time algorithm for computing the Grundy total domination number of a tree. [B,2018]
- For any tree T it holds $\gamma_{\rm gr}^t(T) = 2\beta(T)$, where $\beta(G)$ denotes vertex cover number of a graph G. [B,2018]

[B,2018] B. Brešar, T. Kos, G. Nasini, P. Torres, Total dominating sequences in trees, split graphs, and under modular decomposition, Discrete Optim. 28 (2018) 16–30.

Trees, $\gamma_{gr}^{Z}, \gamma_{gr}^{L}$

- For any tree T it holds $\gamma_{gr}^{L}(T) = |V(T)|$ (which can be shown by linear time algorithm that finds γ_{gr}^{L}-sequence of cardinality $|V(T)|$). [BG,2020]
- For any tree T it holds $Z(T) = P(T)$ ($\gamma_{gr}^{Z}(T) = |V(T)| - P(T)$), where $P(T)$ denotes path cover number of a tree. [Taklimi, 2013]

[BG,2020] B. Brešar, T. G., M.A. Henning and T. Kos. On the L-grundy domination number of a graph, Filomat, 34(10) (2020) 3205–3215.
[Taklimi, 2013] F.A. Taklimi, Zero forcing sets for graphs (Ph.D. thesis), University of Regina, 2013.

Split graphs

A graph G is a split graph, if its vertex set can be partitioned into two subsets, where one subset is a clique and the other is an independent set.

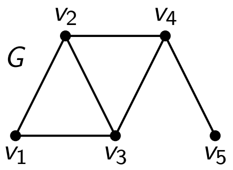
- Grundy domination number of a split graph can be computed in polynomial time. There exists explicit formula that depends on the independence number of the graph. [B,2014]
- Grundy total domination number problem is NP-complete, even when restricted to split graphs. [B,2018]
- L-Grundy domination number problem is NP-complete, even when restricted to split graphs. [BG,2020]

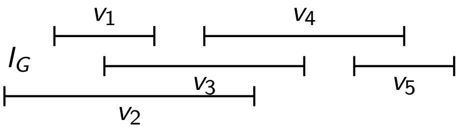
Question: Is there a polynomial time algorithm that computes Z-Grundy domination number of a split graph?

Interval graphs, definitions

- An *interval representation* of a graph is a family of intervals of the real line assigned to vertices so that vertices are adjacent if and only if the corresponding intervals intersect. A graph is an *interval graph* if it has an interval representation.
- Let $G = (V, E)$ be an interval graph with an interval representation $I_G : V(G) \to \{[a, b];\ a, b \in \mathbb{R},\ a \le b\}$, and vertices $V = \{v_1, \dots, v_n\}$ sorted in the non-decreasing order according to the right endpoints of corresponding intervals.
- $I_G(v_i) = [a_i, b_i]$, and $b_1 \le b_2 \le \dots \le b_n$.
- Let $\widehat{A} = \{a_1, b_1, \dots a_n, b_n\}$ be the (multi)set of interval endpoints. We will also make use of the non-decreasing sequence A_{I_G} of the real numbers from $\widehat{A}$ of length $2n$, such that all elements of $\widehat{A}$ are used.

Example





$A_{I_G} = (a_2, a_1, a_3, b_1, a_4, b_2, b_3, a_5, b_4, b_5)$
$S = (v_1, v_2, v_4)$

Figure: An interval graph G with interval representation I_G, interval endpoints sequence A_{I_G} and Grundy dominating sequence S.

Interval graphs, $\gamma_{\rm gr}$

Theorem (BGK,2016)

If G is an interval graph, then $\gamma_{gr}(G)$ equals the number of consecutive subsequences of the form (a_i, b_j) in the interval endpoints sequence A_{I_G} for any interval representation I_G of G.

Theorem (BGK,2016)

If G is an interval graph with vertices $(v_1, v_2, \ldots, v_n)$, ordered according to their right end-points, then $\gamma_{\rm gr}(G)$ can be computed in linear time.

[BGK,2016] B. Brešar, T. G., T. Kos, Dominating sequences under atomic changes with applications in Sierpiński and interval graphs, *Appl. Anal. Discrete Math.* 10 (2016) 518–531.

Interval graphs, $\gamma_{\rm gr}^Z$

For Z-Grundy domination number similar results holds for true twins and simplicial vertices, thus the following problem could be solvable.

Problem

Find an efficient algorithm that returns a $\gamma_{\rm gr}^Z$-sequence of an interval graph.

Extremal graphs

Characterize graphs with

- $\gamma_t(G) = \gamma_{\rm gr}^L(G)$;
- $\gamma_{\rm gr}^t(G) = 2\gamma_{\rm gr}(G)$;
- $\gamma_{\rm gr}^t(G) = 2\gamma_{\rm gr}^Z(G)$;
- $\gamma_{\rm gr}^L(G) = 2\gamma_{\rm gr}(G)$;
- $\gamma_{\rm gr}^t(G) = 2\beta(G)$.

Special graph classes

Consider Grundy domination invariants in

- co-bipartite graphs;
- P_4-tidy graphs;
- Sierpiński graphs;
- Graph products

Special graph classes

Consider Grundy domination invariants in

- co-bipartite graphs;
- P_4-tidy graphs;
- Sierpiński graphs;
- Graph products
  - Conjecture. For any graphs G and H, $\gamma_{\rm gr}(G \boxtimes H) = \gamma_{\rm gr}(G)\gamma_{\rm gr}(H)$.
  - Conjecture. For any graphs G and H, $\gamma_{\rm gr}^t(G \times H) = \gamma_{\rm gr}^t(G)\gamma_{\rm gr}^t(H)$.
  - Both conjectures holds, if G is a tree.

[B,2021] B. Brešar, et al., On Grundy total domination number in product graphs, Discuss. Math. Graph Theory 41.1 (2021) 225–247.
[BD,2021] K. Bell, K. Driscoll, E. Krop, K. Wolff, Grundy domination of forests and the strong product conjecture, Electron. J. Comb. 28(2) (2021) P2.12.

Thank you!