Conflict-free colouring of polygons

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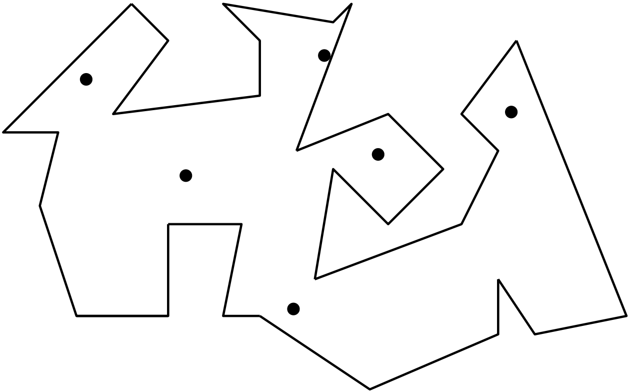
Polygon guarding

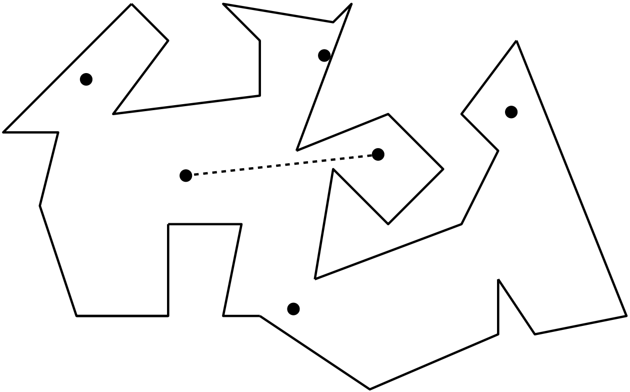
Our objective is to place security cameras or guards in a building so that the building is fully seen.

Polygon guarding



Mathematically, we may consider each floor as a polygon, and the guards as special points that have 360^o vision.

Polygon guarding



Two points are said to be *mutually visible*, or *see each other* if the line segment joining them lies inside the polygon.

Art gallery theorem

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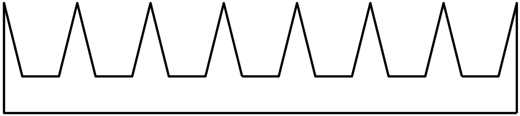
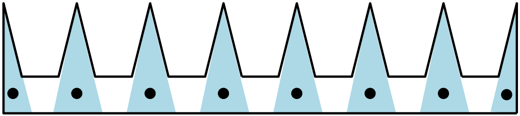
- What is the minimum number of guards sufficient to guard the interior of an n-vertex art-gallery? (Victor Klee, 1973)
- $\lfloor \frac{n}{3} \rfloor$ are sufficient and sometimes necessary to guard any polygon on n vertices. (Chvatal, 1973)

Art gallery theorem



For seeing that $\lfloor \frac{n}{3} \rfloor$ guards are sometimes necessary, consider the above polygon. It is called a comb polygon.

Art gallery theorem

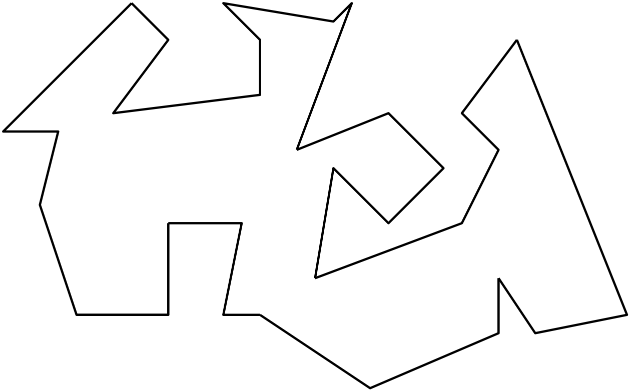


Each tooth of the comb requires a separate guard, thereby making $\lfloor \frac{n}{3} \rfloor$ guards necessary.

Art gallery theorem

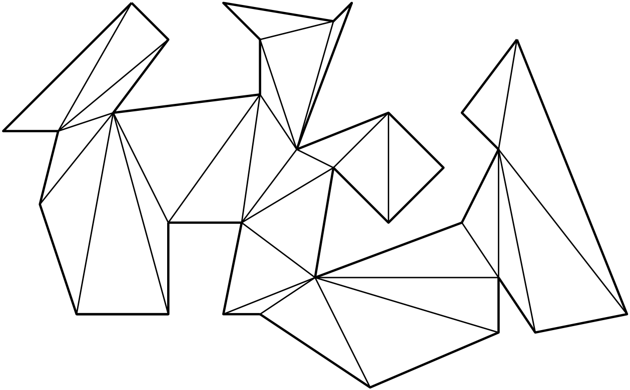
Chvatal also proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient. But his proof is very complicated. So we go for a more elegant proof (Fisk, 1978).

Art gallery theorem



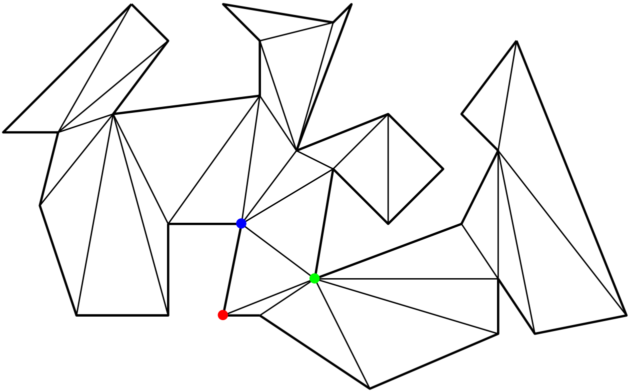
Consider a given polygon.

Art gallery theorem



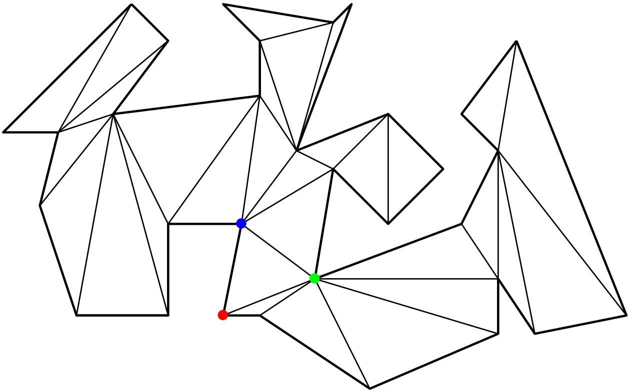
Triangulate the polygon. We are given three colours and our aim is now to colour the vertices of the polygon in such a way that each triangle gets all three colours on its vertices.

Art gallery theorem



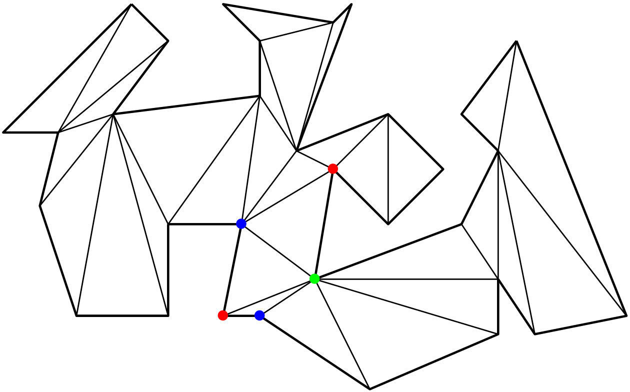
Colour the vertices of any one triangle with three different colours.

Art gallery theorem



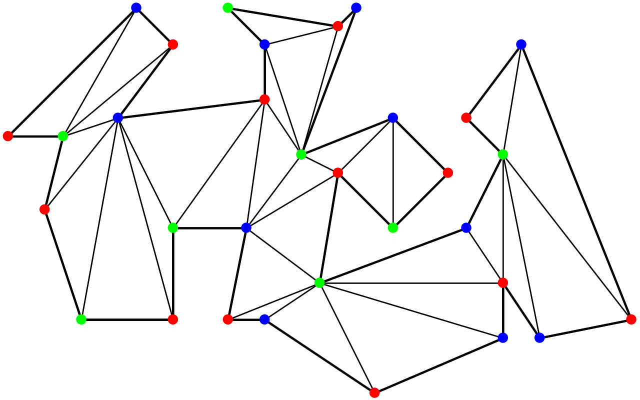
The triangles that share an edge with this triangle already have two out of three of their vertices coloured.

Art gallery theorem



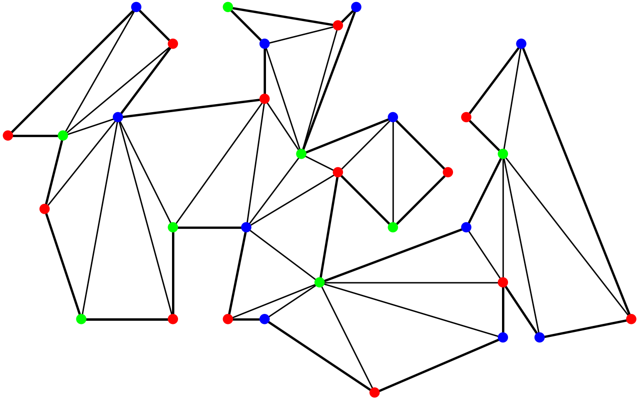
Therefore their remaining vertex can be coloured deterministicaly.

Art gallery theorem



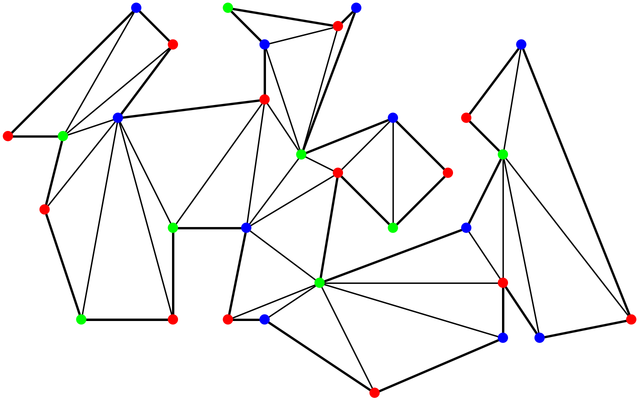
Complete the colouring.

Art gallery theorem



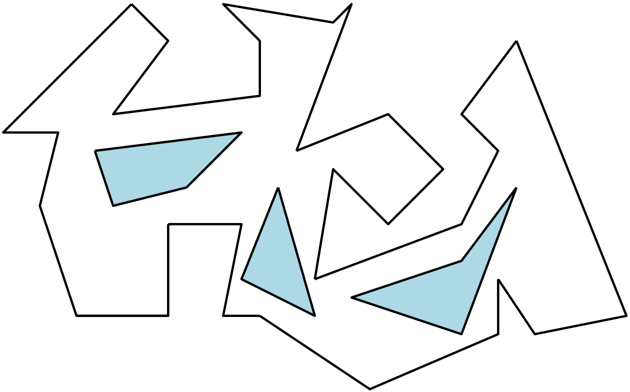
A triangle can be completely seen from any of its 3 vertices. Since each triangle has a red, blue and green vertex, all the vertices of the same colour class see the whole polygon.

Art gallery theorem



By the pigeonhole principle, one of the colour classes has at most $\left\lfloor \frac{n}{3} \right\rfloor$ vertices. We choose that colour class.

Art gallery theorem



Open Problem: Are $\lfloor \frac{n+h}{3} \rfloor$ guards always sufficient for a polygon on n vertices with h holes?

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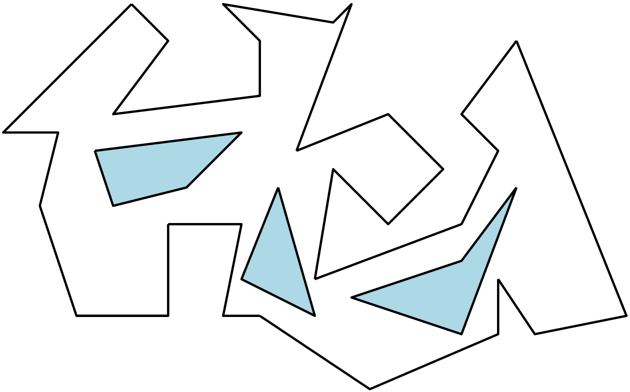

Open Problem: Are $\lfloor \frac{n+h}{3} \rfloor$ guards always sufficient for a polygon on n vertices with h holes? An upper bound of $\lfloor \frac{n+2h}{3} \rfloor$ is known (Alipour, 2021).

Colouring problems

- Graph colouring: Can the vertices of a graph be coloured with k colours such that no two adjacent vertices get the same colour? This is known to be NP-complete.

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- Polygon colouring: Can the vertices of a polygon be coloured with k colours so that no two vertices that see each other get the same colour?

Colouring problems

- For graphs, the colouring problem is polynomial time for k=2, and becomes NP-complete for k=3 onwards (Karp, 1972).

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- Note that if a polygon is 3-colourable, then the Fisk's method produces such a colouring, and otherwise it can be seen that such a colouring does not exist. So the problem is in P for k=3 on polygons.
- Fisk's triangle method can be generalized to show that even 4-colouring is in P for polygons (Cagirici et al, 2017).
- Furthermore, the problem is NP-hard for k=5 on simple polygons and k=4 for polygons with holes (Cagirici et al, 2024).

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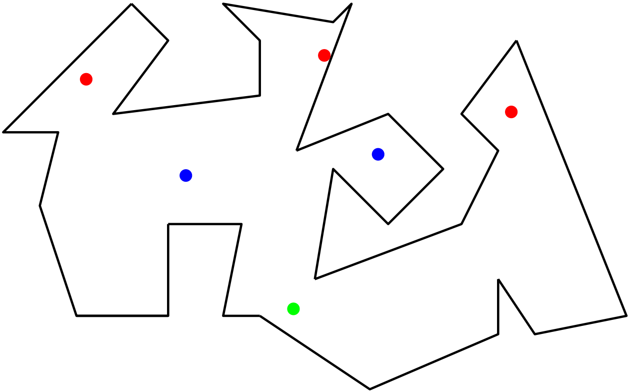
- At each point, the robot must receive signal from at least one transmitter.
- If two different transmitters send signals to the robot in the same frequency, then there will be interference.
- So we need to allot the frequencies so that at each point there is a transmitter that signals the robot in a unique frequency.

Conflict-free colouring

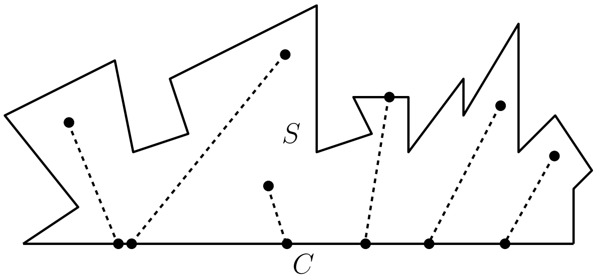


Problem statement: Place guards on the polygon assigning each guard a colour out of k colours, so that every point in the polygon sees a uniquely coloured guard (Guibas et al, 1997).

Conflict-free colouring

We will now see that $O(\log n)$ colours are always sufficient for a polygon on n vertices.

Weak visibility polygon



A weak visibility polygon is a polygon S with an edge C such that each point in S is visible from some point in C. Note that the point on C can differ with choice of the point of S.

Polygon decomposition

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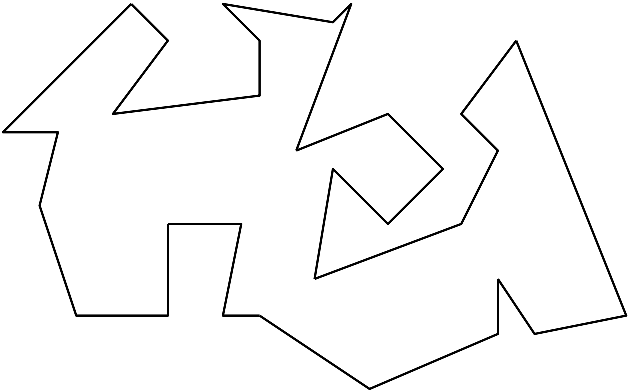
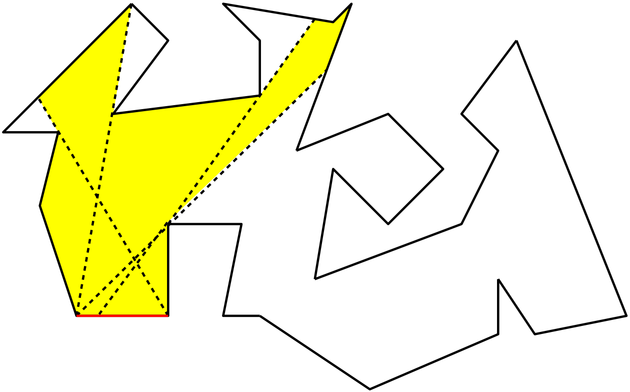
- Finally, we will prove the bound for a weak visibility polygon.

Polygon decomposition



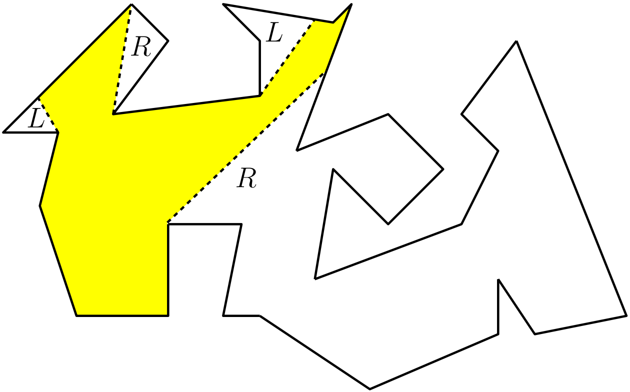
Consider any given polygon, and three colour classes, yellow, blue and green. Each colour class has $O(log\ n)$ colours.

Polygon decomposition



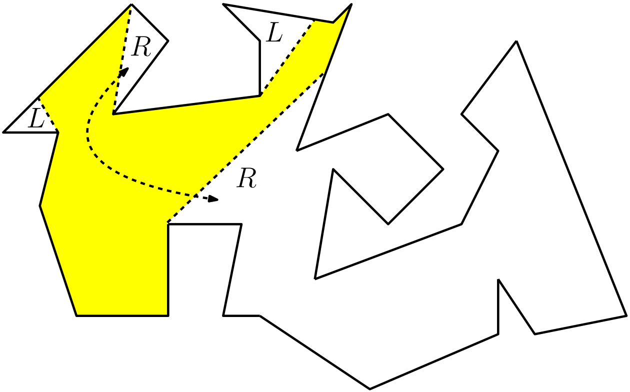
Consider an edge, construct its visibility polygon and colour it with the yellow colour class (assume for the time being that we can do this).

Polygon decomposition



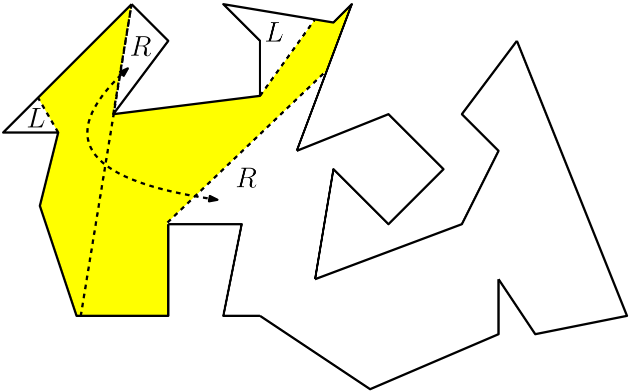
The remaining polygon can be divided into parts that are to the right or left of the constructed edges of this weak visibility polygon.

Polygon decomposition



Observe that no point in a right region can see any point of another right region. The same holds for left regions as well.

Polygon decomposition



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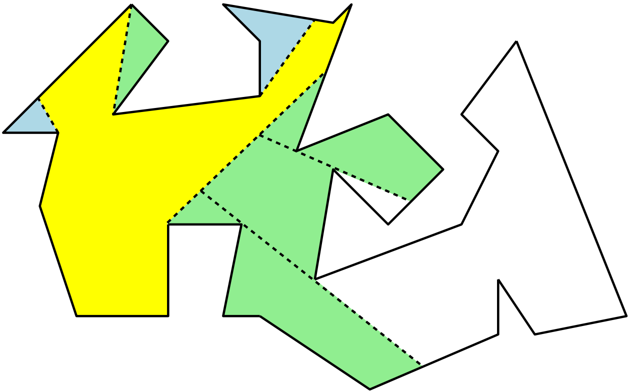
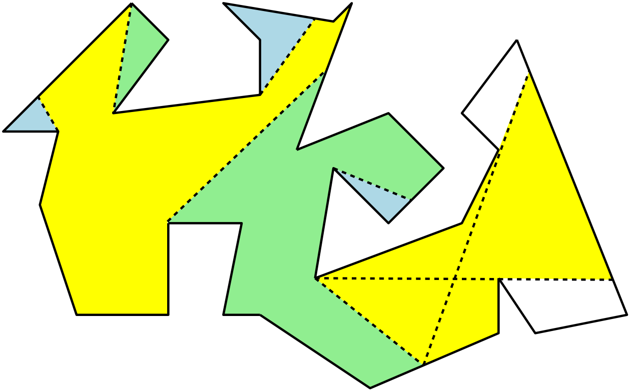
Polygon decomposition



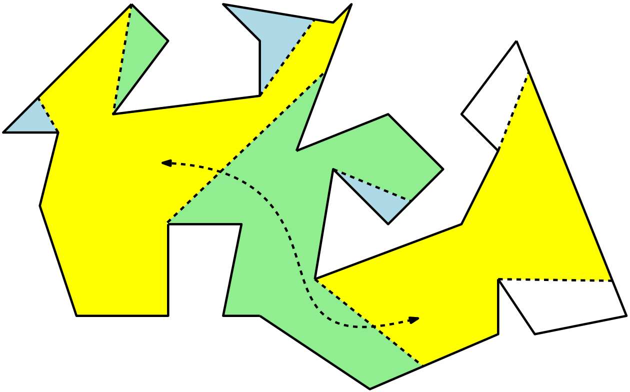
So we can construct the weak visibility polygons on these constructed edges, and assign the same colour class to the ones in right regions, and the other colour class to the ones in left regions.

Polygon decomposition



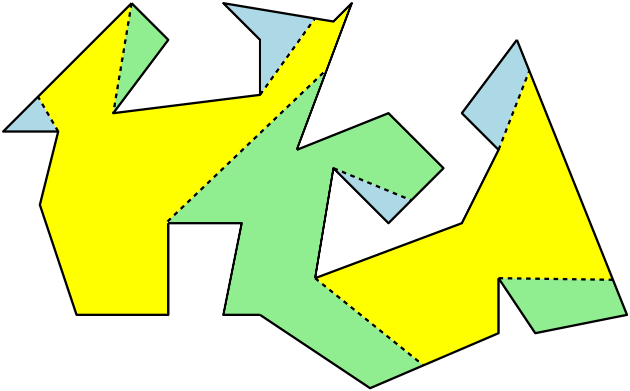
We repeat the process from the new constructed edges and draw the next layer of weak visibility polygons.

Polygon decomposition



Observe that no points from these third layer weak visibility polygons can see the original weak visibility polygon.

Polygon decomposition



Hence we can complete the colouring so that no two weak visibility polygons assigned the same colour class can see any point of each other.

Polygon decomposition

Thus, if we can produce a conflict free colouring of a weak visibility polygon with $O(log\ n)$ colours, then we can do so for any polygon.

Ruler sequence

$$1 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 4 \quad 1 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 5 \ldots$$

The above sequence is called the ruler sequence.

Ruler sequence

$$\boxed{1\quad 2\quad 1\quad 3\quad 1\quad 2\quad 1\quad 4}\ \boxed{1\quad 2\quad 1\quad 3\quad 1\quad 2\quad 1\quad 5\ldots}$$

Let n be a power of 2. Denote by S_n the ruler sequence up to the n^{th} term. S_n is obtained by concatenating two copies of $S_{n/2}$ and increasing the last term of the second copy by 1.

Ruler sequence

$$\boxed{1 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 4} \; \boxed{1 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 5 \ldots}$$

Due to this method of construction, the highest value in the ruler sequence up to n terms is only around $\sim log\ n$.

Ruler sequence

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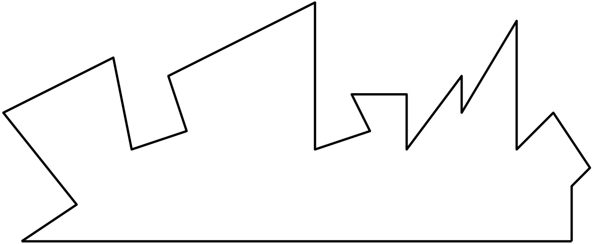
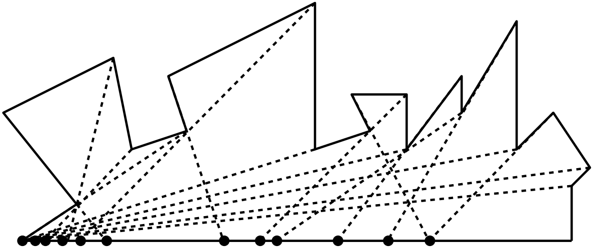
Also, any subsequence of consecutive terms always has a unique term (prove by induction).

Conflict-free colouring a weak visibility polygon



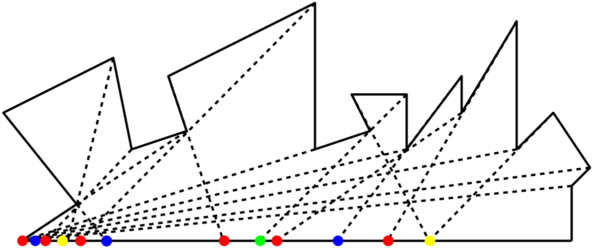
Consider any given weak visibilty polygon.

Conflict-free colouring a weak visibility polygon



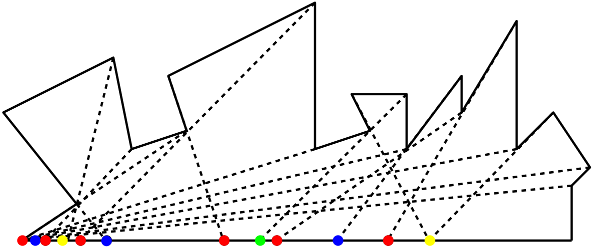
On the weak visibility edge, mark the leftmost point that each vertex sees.

Conflict-free colouring a weak visibility polygon



Colour these points according to the ruler sequence.

Conflict-free colouring a weak visibility polygon



This is a conflict-free colouring since every point of the polygon sees a continuous segment of the weak visibility edge.

Conflict-free colouring a polygon

- Thus, any polygon on n vertices can be conflict-free coloured with $O(log\ n)$ colours when the coloured points are allowed to be placed anywhere in the polygon (Bärtschi et al, 2014).

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Conflict-free colouring a polygon

- Thus, any polygon on n vertices can be conflict-free coloured with $O(log\ n)$ colours when the coloured points are allowed to be placed anywhere in the polygon (Bärtschi et al, 2014).
- Open Problem: Find a lower bound for the problem.
- Open Problem: Is conflict-free colouring a polygon NP-hard?

Conflict-free colouring a polygon

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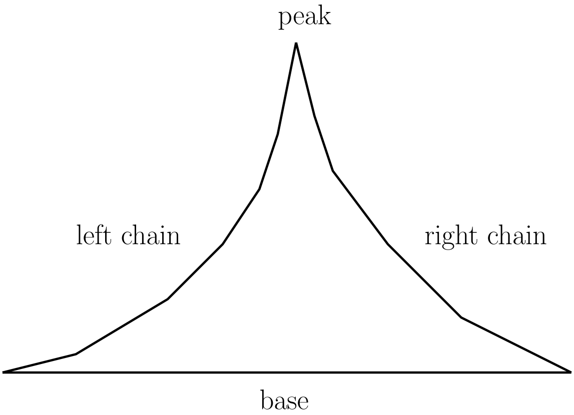
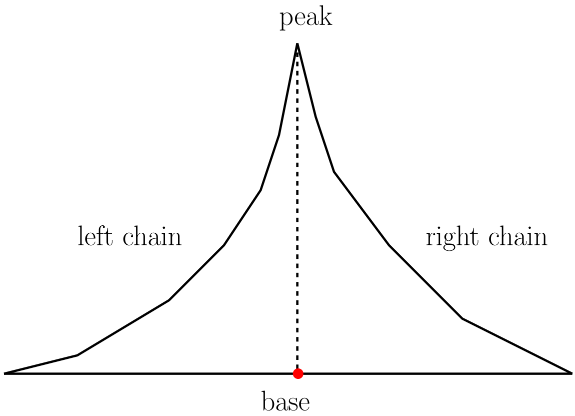
- The polygon decomposition method will still work with minor modifications.
- So we will discuss an algorithm for weak visibility polygons.

Funnels



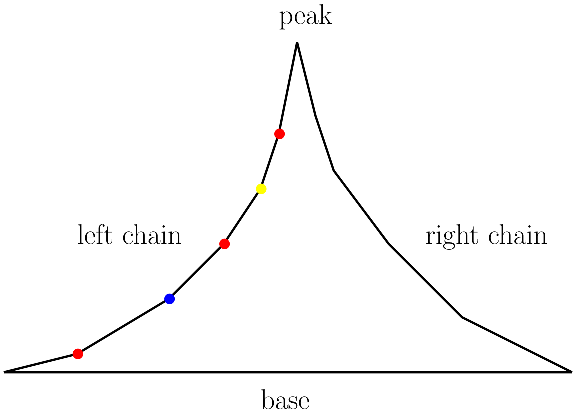
This type of polygon with a base, a peak and two concave chains from the base to the peak, is called a *funnel*.

Funnels



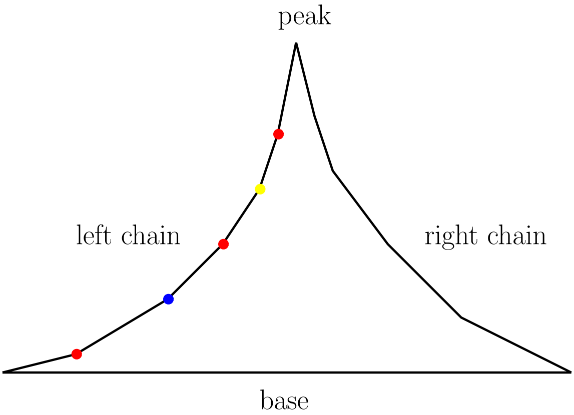
When the coloured points are allowed to be placed anywhere, a funnel can be conflict-free coloured using only one point; the point on the base vertically below the peak.

Funnels



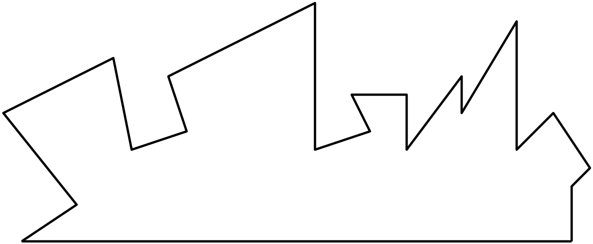
When the coloured points are restricted to vertices, the ruler sequence comes to our rescue!

Funnels



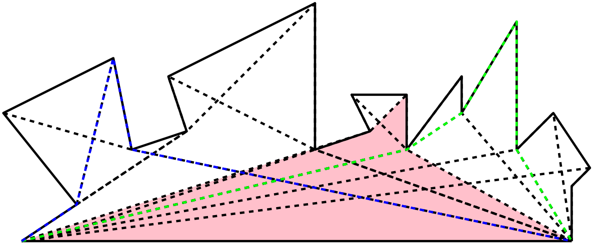
Thus, a funnel can be conflict-free coloured by $O(log\ n)$ colours even when the guards are restricted to vertices (V2P version).

V2P conflict-free colouring



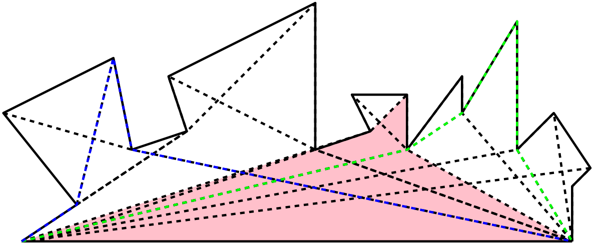
A weak visibility polygon can be considered as a union of multiple funnels on the same base.

V2P conflict-free colouring



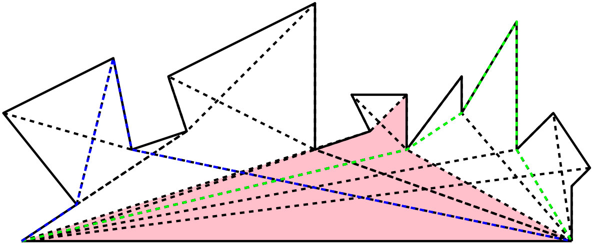
A weak visibility polygon can be considered as a union of multiple funnels on the same base plus some extra regions between the funnels, such that the funnels use up all the vertices.

V2P conflict-free colouring



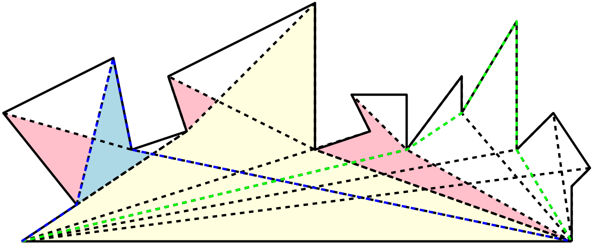
Consider *log n* colour classes, each with *log n* distinct colours.

V2P conflict-free colouring



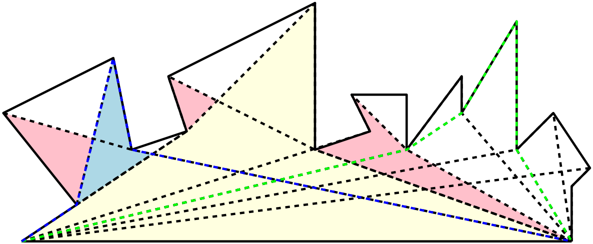
Starting from the left base vertex, traverse the boundary of the polygon in clockwise order and consider funnels in the order in which we encounter their peaks.

V2P conflict-free colouring



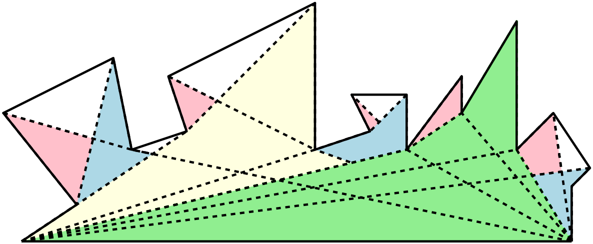
In the same order, assign colour classes to the funnels according to the ruler sequence.

V2P conflict-free colouring



Vertices get colours according to the ruler sequence applied on each colour class.

V2P conflict-free colouring



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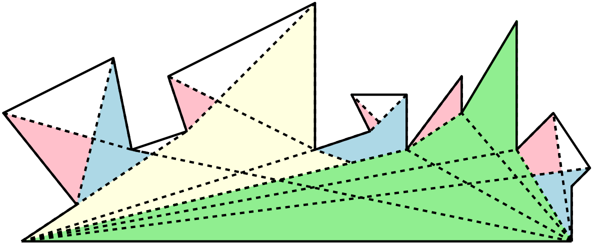
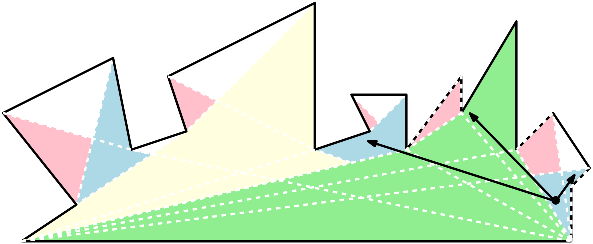


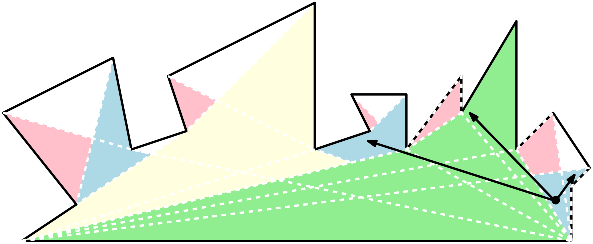
If a vertex belongs to multiple funnels, it gets its colour from the colour class corresponding to the higher term in the ruler sequence.

V2P conflict-free colouring



Due to this, if a point sees vertices from two different funnels assigned the same colour class, it also sees a vertex from an intermediate funnel assigned a higher colour class.

V2P conflict-free colouring



There can be at most n funnels, so we need at most $log\ n$ colour classes. Since each class contains $log\ n$ colours, we need $log^2\ n$ colours in total.

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- Open Problem: Is the V2P conflict-free colouring problem NP-hard?

Thank you!