Frobenius Problem for the Proth numbers

Joint work with Dhara Thakkar (IIT Gandhinagar)

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Indian Institute of Technology Bhilai

Plan for the Talk

- Frobenius Problem
- Frobenius Problem for special numbers
- Wilf Conjecture

The Frobenius Problem:

Given: A set $L = \{l_1, l_2, \ldots, l_m\}$ with $\gcd(l_1, \ldots, l_m) = 1$, and $l_i \geq 2$.

Question: Find the largest natural number that is not expressible as a non-negative linear combination of $l_1, l_2, ..., l_m$.

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Other Names:

- The Money Exchange Problem
- The Chicken Nuggets Problem

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Claim : 43 is not expressible using $6, 9, 20$

- We can choose ≤ 2 packs of 20.
- If we choose 0 or 1 packs, then we have to represent 43 or 23 as a linear combination of 6 and 9. Not Possible!
- If we choose two packs of 20 then we can not represent 3 using of 6 and 9. Again Not Possible!

To see that every larger number is expressible, note that

$$
\begin{aligned}
44 &= 1 \cdot 20 + 0 \cdot 9 + 4 \cdot 6 \\
45 &= 0 \cdot 20 + 3 \cdot 9 + 3 \cdot 6 \\
46 &= 2 \cdot 20 + 0 \cdot 9 + 1 \cdot 6 \\
47 &= 1 \cdot 20 + 3 \cdot 9 + 0 \cdot 6 \\
48 &= 0 \cdot 20 + 0 \cdot 9 + 8 \cdot 6 \\
49 &= 2 \cdot 20 + 1 \cdot 9 + 0 \cdot 6
\end{aligned}
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and every larger number can be written as a multiple of 6 plus one of these numbers.

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  - **Several special cases**, e.g., numbers in a geometric sequence, arithmetic sequence, Pythagorean triples, three consecutive squares or cubes, and many more!
  - The Frobenius problem has been studied for **several special numerical semigroups** that naturally arises from special prime like Fibonacci, Mersenne, Thabit, and Repunit.

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Example: $S = \{6, 9, 12, 15, 18, 20, 24, \ldots, 42, 44 \rightarrow\}$ and $\mathbb{N} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 37, 43\}$.

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Definition: The Frobenius number ($\mathrm{F}(S)$) of a numerical semigroup $S = \langle \{a_1, a_2, \ldots, a_n\} \rangle$ is the largest integer that cannot be expressed as a sum $\sum_{i=1}^{n} t_i a_i$, where $t_1, \ldots, t_n \in \mathbb{N}$.

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The Proth Numerical semigroups

Definition: The Proth number is a natural number of the form $k2^n + 1$, where $n, k \in \mathbb{Z}^+$ and $k < 2^n$ is an odd number. E.g., 3, 5, 9, 13, 17, 25, 33, 41, 49.

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Definition: A numerical semigroup S is the Proth numerical semigroup if $n \in \mathbb{N}$ such that

$$S = \langle \{k2^{n+i} + 1 \mid i \in \mathbb{N}\} \rangle,$$

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Why Proth Numerical Semigroup?

Surprisingly, the methods that has been used to study the Frobenius Problem for the Fibonacci, Mersenne, Thabit, Repunit numerical semigroup is not *directly* applicable to the Proth Numerical semigroup.

Embedding Dimension of the Proth Numerical Semigroup

Theorem 1 [S,Thakkar]

Let $n > 2$ be an integer then

$$\mathrm{e}(P_k(n)) = n + r + 1.$$

Moreover, $\left\{k2^{n+i} + 1 \mid i \in \{0, 1, \ldots, n + r\}\right\}$ is the minimal system of generators of $P_k(n)$.

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Notation: We now take $s_i = k2^{n+i} + 1$ for all $i \in \mathbb{N}$. Thus, with this notation, $\{s_0, s_1, ..., s_{n+r}\}$ is the minimal system of generators of $P_k(n)$.

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Lemma [Selmer77] Let S be a numerical semigroup and let s' be a non-zero element of S. Then
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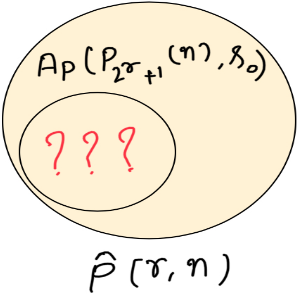

Elements that are Not in the Apéry set

Lemma [S, Thakkar] Let $n > 2$ be an integer. Then

$$F \cap \mathrm{Ap}(P_{2^r+1}(n), s_0) = \emptyset,$$

where $F = F_1 \cup F_2$, and

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$$F_2 = \Big(\bigcup_{l=0}^{r-2} E_l \cup \{2s_{n+r}\}\Big) \setminus \{s_1 + s_n + s_{n+r},\ 2s_1 + s_n + s_{n+r},\ s_n + s_{n+r}\},$$

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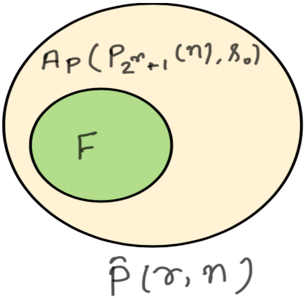

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- $\mathrm{Ap}(P_{2^r+1}(n), s_0) \subseteq \hat{P}(r, n)$.
- $F \cap \mathrm{Ap}(P_{2^r+1}(n), s_0) = \emptyset$.
- $|\mathrm{Ap}(P_{2^r+1}(n), s_0)| = s_0$,
  $|F| = 2^{n+r} - 2^n - 2$, and
  $|\hat{P}(r, n)| = 2^{n+r+1} - 1$.

Theorem: [S, Thakkar] Let $n > 2$ be an integer. Then

$$\mathrm{Ap}(P_{2^r+1}(n), s_0) = \{a_1 s_1 + \cdots + a_{n+r} s_{n+r} \mid (a_1, \ldots, a_{n+r}) \in P(r,n)\} \setminus F.$$

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The Frobenius Number

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Lemma [S, Thakkar]

Let $s \in P_{2^r+1}(n)$ such that $s \not\equiv 0 (\bmod s_0)$, then $s + 1 \in P_{2^r+1}(n)$. Moreover,

- $w(i+1) \leq w(i) + 1$ for $1 \leq i \leq s_0 - 1$.
- $w(2) = s_1 + s_n + s_{n+r}$;
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Theorem [S, Thakkar]

Let $n > 2$ be a positive integer. Then the Frobenius number of the Proth numerical semigroup is given by

$$\mathrm{F}(P_{2^r+1}(n)) = 2s_1 + s_n + s_{n+r} - s_0.$$

Wilf Conjecture

Wilf Conjecture [Wilf78]

Let S be a numerical semigroup, and $\nu(S) = |\{s \in S \mid s \leq \mathrm{F}(S)\}|$, then

$$\mathrm{F}(S) + 1 \leq \mathrm{e}(S)\nu(S),$$

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- Almost arithmetic sequence.
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Theorem [S, Thakkar]

The Proth numerical semigroup $P_{2^r+1}(n)$ satisfies Wilf's conjecture.

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Definition: An integer x is a pseudo-Frobenius number of S if $x \in \mathbb{Z} \setminus S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$.

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Consider the relation on $\mathbb{Z}$: $a \leq_S b$ if $b - a \in S$. Then $\leq_S$ is is an order relation.

$$\mathrm{PF}(S) = \{w - s' \mid w \in \mathrm{maximals}_{\leq_S}(\mathrm{Ap}(S, s'))\}$$

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Definition: The cardinality of the set $\mathrm{PF}(S)$ is called the type of S denoted by $\mathrm{t}(S)$

Theorem [S, Thakkar] Let $n > 2$ be an integer and let $P_{2^r+1}(n)$ be the Proth numerical semigroup. Then

$$\begin{aligned}\mathrm{PF}(P_{2^r+1}(n)) = \{&2s_i + s_{i+1} + \cdots + s_{n+r-1} - s_0 \mid 1 \leq i \leq r\} \cup \\ &\{2s_j + s_{j+1} + \cdots + s_{n-1} + s_{n+r} - s_0 \mid 1 \leq j \leq n-2\} \\ &\cup \{2s_1 + s_n + s_{n+r} - s_0\}, \text{and} \\ \mathrm{t}(P_{2^r+1}(n)) = |\mathrm{PF}(P_{2^r+1}(n)| &= r + n - 1.\end{aligned}$$

Lemma [**ADG20**]) Let S be a numerical semigroup. We have

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$$\begin{aligned} \mathrm{F}(P_{2^r+1}(n)) + 1 &\leq (\mathrm{t}(P_{2^r+1}(n)) + 1)\,\nu(P_{2^r+1}(n)) \\ &= (n + r)\,\nu(P_{2^r+1}(n)) \\ &< (n + r + 1)\,\nu(P_{2^r+1}(n)) \\ &= \mathrm{e}(P_{2^r+1}(n))\,\nu(P_{2^r+1}(n). \end{aligned}$$

□

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Thank you!