Open Packing in H-free Graphs and Some Subclasses of Split Graphs

M. A. Shalu V. K. Kirubakaran

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Outline

Introduction

Definitions

Total Dominating Set

A set $D \subseteq V(G)$ is called a total dominating set of G if for vertex $u \in V(G)$, there exists a vertex $x \in D$ such that $xu \in E(G)$
i.e., $|N(u) \cap D| \geq 1$ for every $u \in V(G)$.

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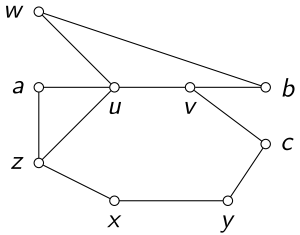

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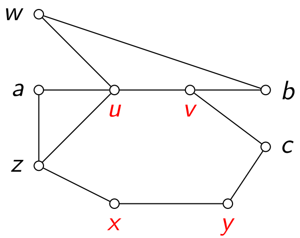

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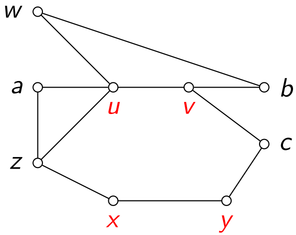

A graph G with a total dominating set $D = \{u, v, x, y\}$ and total domination number, $\gamma_t(G) = 4$.

Definitions

Open Packing

A set $S \subseteq V(G)$ is called an open packing in G if no two distinct vertices in S have a common neighbour in G

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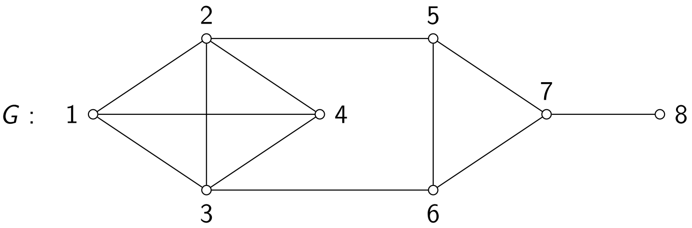

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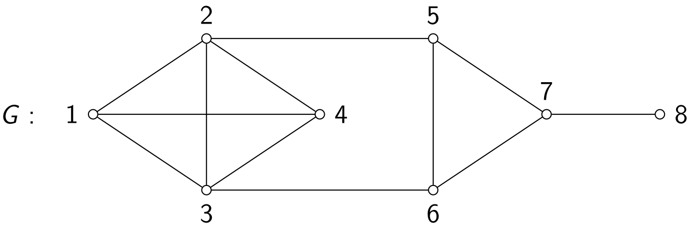

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Given a graph G and a vertex subset D of G,

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$$\implies \gamma_t(G) \geq \rho^o(G)$$

Computational Problems

Total Dominating Set (TDS)

Instance: A graph $G(V, E)$ and a positive integer $k \leq |V(G)|$.
Question: Is there a total dominating set of size k in G?

Open Packing

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MIN-TOTAL DOMINATING SET

Instance : A graph G.
Task : Find $\gamma_t(G)$.

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MAX-OPEN PACKING

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- Antony et al. (2023) proved that Total Dominating Set is NP-complete for r-regular triangle-free graphs for every $r \geq 3$.

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- Relation between open packing number and other graph parameters such as P_3-radon number are also studied in the literature [Henning et al. (2013)].
- 3-independent set, subclique and injective coloring are some of the graph related to open packing.

H-free Graphs

Results

(1.) OPEN PACKING is NP-complete on $K_{1,3}$-free graphs.

(2.) For every $r \geq 1$ and for every connected $(P_4 \cup rK_1)$-free graph G, $\rho^o(G) \leq 2r + 1$ (This bound is tight).

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The above set of results eventually imply that

Theorem 1

For $p \geq 4$, let H be a graph on p vertices. Then, Open Packing *is polynomial time solvable on the class of H-free graphs if and only if $H \in \{pK_1, (K_2 \cup (p-2)K_1), (P_3 \cup (p-3)K_1), (P_4 \cup (p-4)K_1)\}$ unless $NP = P$.*

Observation 1

For $p \geq 4$, let H be a graph on p vertices such that $H \notin \{P_4 \cup (p-4)K_1, P_3 \cup (p-3)K_1, K_2 \cup (p-2)K_1, pK_1\}$. Then, H contains H' for some $H' \in \{K_3, 2K_2, C_4, K_{1,3}, C_5\}$ as an induced subgraph.

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Case 1: $H' = K_3$.

Bipartite Graphs $\subseteq$ K_3-free graphs

Open Packing is NPC in bipartite graphs [1,2]	$\Longrightarrow$	Open Packing is NPC in K_3-free graphs.

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Split Graphs $\subseteq$ H'-free graphs.

OPEN PACKING is NPC in split graphs [3] $\Longrightarrow$ OPEN PACKING is NPC in (i) $2K_2$-free graphs, (ii) C_4-free graphs and (iii) C_5-free graphs.

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So, if H contains $H' \in \{K_3, 2K_2, C_4, C_5\}$, then OPEN PACKING is NP-complete in H-free graphs

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$K_{1,3}$-free Graphs

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Open Packing *is NP-complete for* $K_{1,3}$*-free graphs.*

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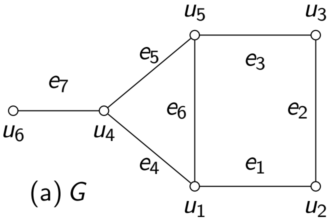
INDEPENDENT SET
Instance: A simple graph G and a positive integer $k \leq \lvert V(G) \rvert$.
Question: Does G contains an independent set of size k?

Karp (1972) proved that INDEPENDENT SET is NP-complete for simple graphs.

$K_{1,3}$-free Graphs

Construction 1

Input: A simple graph G with $V(G) = \{u_1, u_2, \ldots, u_n\}$.
Output: A $K_{1,3}$-free graph G'.



(a) G

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Input: A simple graph G with $V(G) = \{u_1, u_2, \ldots, u_n\}$.
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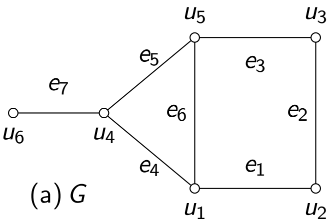
Gaurantee: G has an independent set of size k if and only if G' has an open packing of size k.



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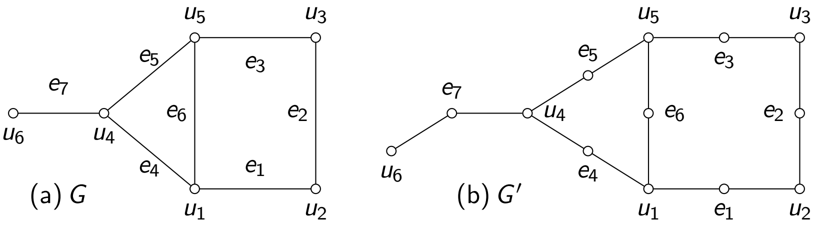

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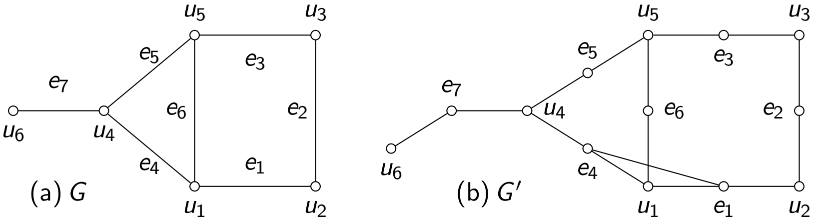

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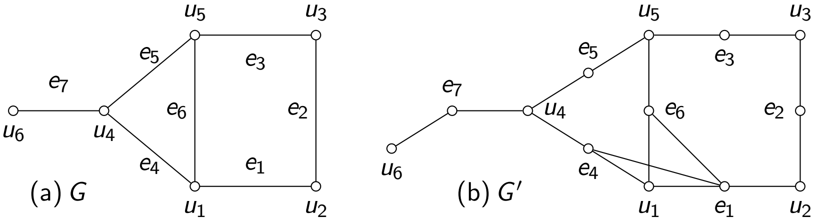

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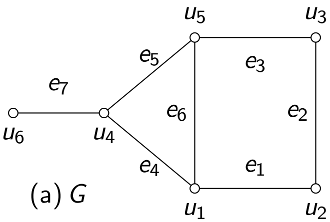

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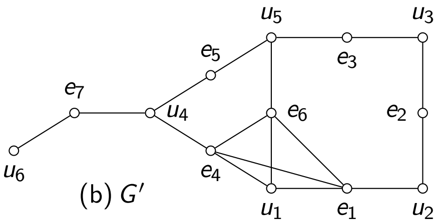

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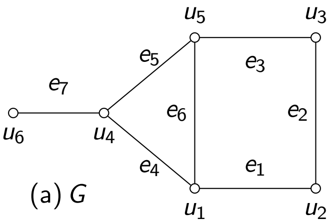

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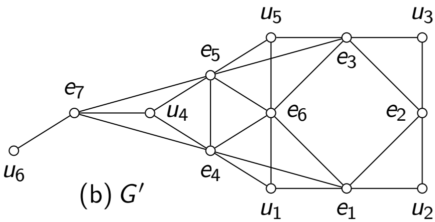

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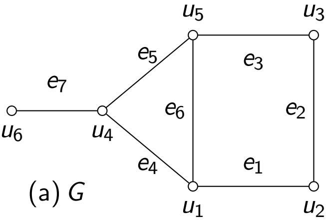

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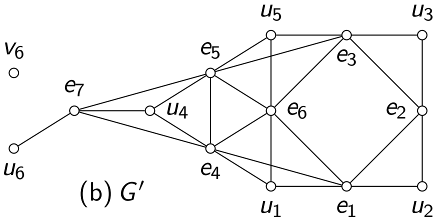

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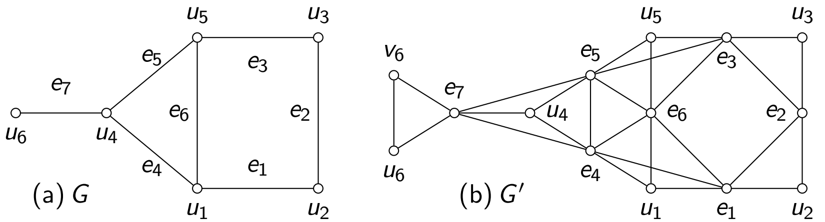

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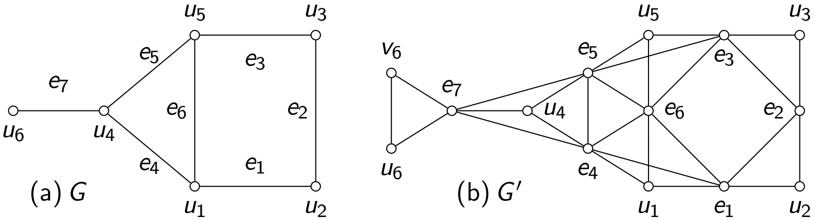

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$K_{1,3}$-free Graphs

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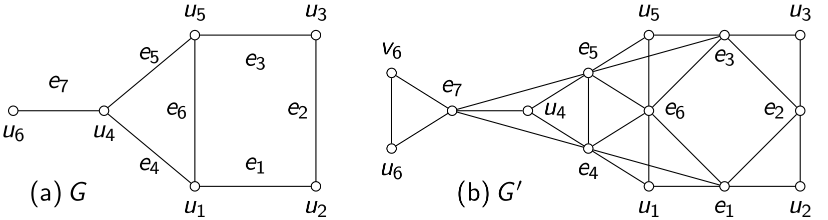
The graph G' is $K_{1,3}$-free.
Assume the contrary, that G' has a $K_{1,3}$ with some vertex $x \in V(G') = V(G) \cup E(G) \cup \{v_i : 1 \leq i \leq n, d_G(u_i) = 1\}$ as centre.



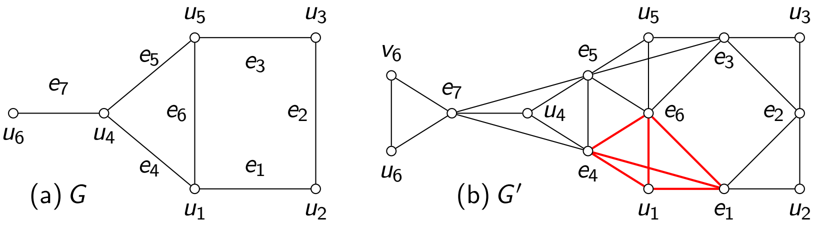
(a) G (b) G'

$K_{1,3}$-free Graphs

Construction 1

The graph G' is $K_{1,3}$-free.
Assume the contrary, that G' has a $K_{1,3}$ with some vertex $x \in V(G') = V(G) \cup E(G) \cup \{v_i : 1 \leq i \leq n, d_G(u_i) = 1\}$ as centre.

- *$x \notin V(G) \cup \{v_i : 1 \leq i \leq n, d_G(u_i) = 1\}$ since $N_{G'}(u_i)$ and $N_{G'}(v_i)$ is a clique.*



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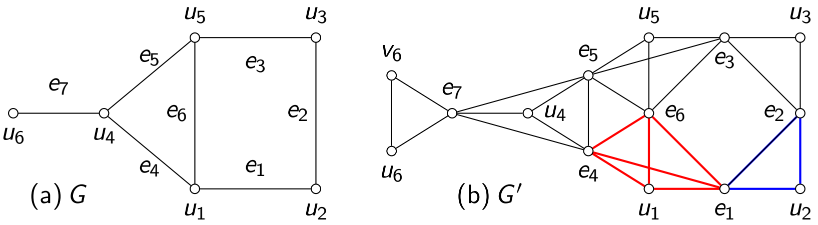
- *$x \notin V(G) \cup \{v_i : 1 \leq i \leq n, d_G(u_i) = 1\}$ since $N_{G'}(u_i)$ and $N_{G'}(v_i)$ is a clique.*
- *$x \notin E(G)$ because for all $e = uu' \in E(G)$, $N_{G'}[e] = N_{G'}[u] \cup N_{G'}[u']$ is a union of two cliques.*

$\implies$ *no such x exists which is a contradiction.*



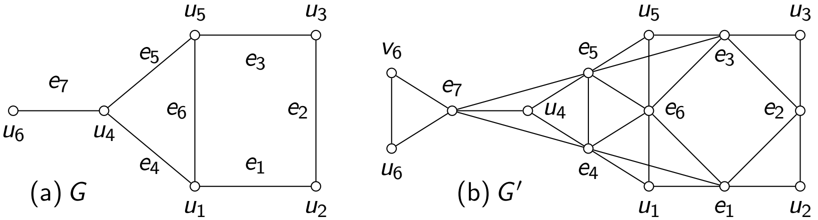
(a) G (b) G'

$K_{1,3}$-free Graphs

Construction 1

Idea:

$\Longrightarrow$ $S \subseteq V(G)$ *is an independent set in* G *if and only if* S *is an open packing in* G'. *(i.e.,* $uv \notin E(G) \iff N_{G'}(u) \cap N_{G'}(v) = \emptyset$ *)*



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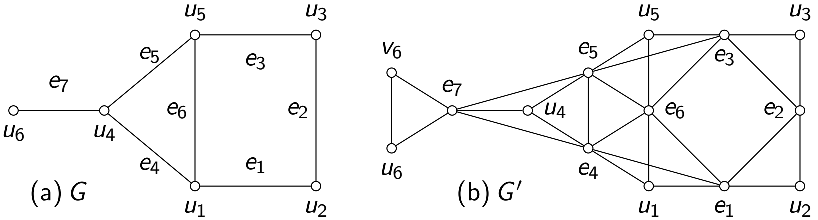

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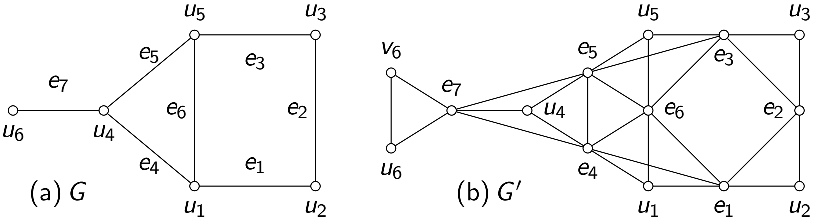
$\Longleftarrow$ *Suppose S is an open packing in G'.*

1. *If $v_i \in S$, then replace v_i by u_i in S.*



$K_{1,3}$-free Graphs

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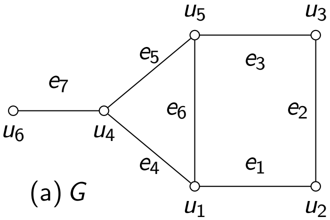
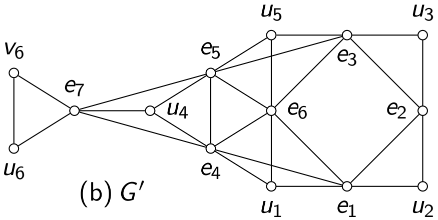
$\Longleftarrow$ *Suppose S is an open packing in G'.*

1. *If $v_i \in S$, then replace v_i by u_i in S.*
2. *If $e = uu' \in S$, then no vertex having a common neighbour with u or u' in G' is in S. Replace e either with u or u' in S.*

This completes the Guarantee of Construction 1.



(a) G



(b) G'

$K_{1,3}$-free Graphs

Theorem 4

MAX-OPEN PACKING *is hard to approximate within a factor of* $N^{\frac{1}{2}-\epsilon}$ *for any* $\epsilon > 0$ *in* $K_{1,3}$*-free graphs unless* $NP = P$ *where* N *denotes the number of vertices in a* $K_{1,3}$*-free graph.*

$K_{1,3}$-free Graphs

Theorem 3 (Håstard (1999))

MAX-INDEPENDENT SET *cannot be approximated within a factor of* $n^{(1-\epsilon)}$ *for any* $\epsilon > 0$*, in general graphs unless NP=P.*

Theorem 4

MAX-OPEN PACKING *is hard to approximate within a factor of* $N^{\frac{1}{2}-\epsilon}$ *for any* $\epsilon > 0$ *in* $K_{1,3}$*-free graphs unless* $NP = P$ *where* N *denotes the number of vertices in a* $K_{1,3}$*-free graph.*

Theorem 4 follows from Theorem 3 and Construction 1.

$K_{1,3}$-free Graphs

Theorem 7

Open Packing *parameterized by solution size is W[1]-complete on* $K_{1,3}$*-free graphs.*

$K_{1,3}$-free Graphs

Lemma 5 (Rall (2005))

Given a graph G, let the neighbourhood graph $G^{[o]}$ of G be a simple graph with $V(G^{[o]}) = V(G)$ and $E(G^{[o]}) = \{xy : x, y \in V(G), x \neq y \text{ and } N_G(x) \cap N_G(y) \neq \emptyset\}$. Then, a vertex subset S is an open packing in G if and only if S is an independent set in $G^{[o]}$.

Theorem 6 (Downey and Fellows (1995))

INDEPENDENT SET *parameterized by solution size is W[1]-complete on simple graphs.*

Theorem 7

OPEN PACKING *parameterized by solution size is W[1]-complete on $K_{1,3}$-free graphs.*

Theorem 7 follows from Theorem 6, Lemma 5 and Construction 1.

Observation 1

For $p \geq 4$, let H be a graph on p vertices such that $H \notin \{P_4 \cup (p-4)K_1, P_3 \cup (p-3)K_1, K_2 \cup (p-2)K_1, pK_1\}$.
Then, H contains one of $K_3, 2K_2, C_4, K_{1,3}$ or C_5 as an induced subgraph.

Known: Open Packing is NP-complete for (i) K_3-free graphs, (ii) $K_{1,3}$-free graphs, (iii) C_4-free graphs, (iv) $2K_2$-free graphs and (v) C_5-free graphs.

Sufficiency Part of Theorem 1

For a graph H on p vertices with $p \geq 4$, Open Packing is polynomial time solvable in H-free graphs only if
$H \in \{P_4 \cup (p-4)K_1, P_3 \cup (p-3)K_1, K_2 \cup (p-2)K_1, pK_1\}$ unless $NP = P$.

Necessary Part of Theorem 1

For $p \geq 4$, if $H \in \{P_4 \cup (p-4)K_1, P_3 \cup (p-3)K_1, K_2 \cup (p-2)K_1, pK_1\}$, then Open Packing is polynomial time solvable in H-free graphs.

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Note that (i) $(P_3 \cup (p-3)K_1)$-free graphs $\subseteq (P_4 \cup (p-3)K_1)$-free graphs

(ii) $(K_2 \cup (p-2)K_1)$-free graphs $\subseteq (P_4 \cup (p-3)K_1)$-free graphs

(iii) pK_1-free graphs $\subseteq (P_4 \cup (p-2)K_1)$-free graphs

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(iii) pK_1-free graphs $\subseteq (P_4 \cup (p-2)K_1)$-free graphs

$\Longrightarrow$ To prove the necessary part, it is enough to show that OPEN PACKING $\in$ P in the class of $(P_4 \cup rK_1)$-free graphs for every $r \geq 0$.

$(P_4 \cup rK_1)$-free Graphs

Lemma 8

Given a graph class $\mathcal{G}$, if there exists $k \in \mathbb{N}$ such that $\rho^o(G) \leq k$ for every $G \in \mathcal{G}$, then
(i) G contains at most $O(n^k)$ open packings and
(ii) all open packings in G can be computed in $O(n^{k+1})$ time for every $G \in \mathcal{G}$.
So, $\rho^o(G)$ can be computed in $O(n^{k+1})$ time.

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Lemma 9 (Folklore)

For connected P_4-free graphs, $\rho^o(G) \leq \gamma_t(G) = 2$.

Lemma 10

For $r \geq 1$, if G is a connected $(P_4 \cup rK_1)$-free graph, then $\rho^o(G) \leq 2r + 1$.

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$$k = 2r + 1$$

Remark 1

The bound given in Lemma 10 is tight. An example for the case $r = 3$ is given below.

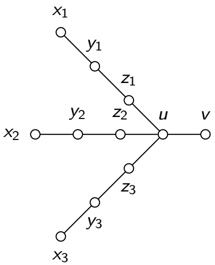

Figure: A $(P_4 \cup 3K_1)$-free graph G_3

Remark 1

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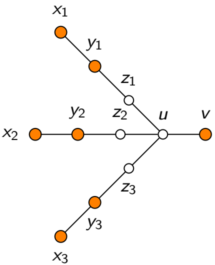

Figure: A $(P_4 \cup 3K_1)$-free graph G_3 with an open packing $S_3 = \{x_1, x_2, x_3, y_1, y_2, y_3, v\}$ of size $7 = (2(3) + 1)$.

Theorem 1

For $p \geq 4$, let H be a graph on p vertices. Then, Open Packing is polynomial time solvable on the class of H-free graphs if and only if $H \in \{pK_1, (K_2 \cup (p-2)K_1), (P_3 \cup (p-3)K_1), (P_4 \cup (p-4)K_1)\}$ unless $NP = P$.

Proved!

Subclasses of Split Graphs

Objective

Complexity comparision between Total Dominating Set and Open Packing

Graph Class	Total Dominating Set	Open Packing
Chordal Bipartite Graphs	P [4]	P [2]
H-free Graphs	P/NP	P/NP
Bipartite Graphs	NPC [5]	NPC [2,6]
Split Graphs	NPC [7]	NPC [3]

[2] - Shalu and Kirubakaran (2023)
[3]- Ramos et al. (2014)
[4] - Damaschke et al. (1990)
[5] - Pfaff et al. (1983)
[6] - Shalu et al. (2017)
[7] - Corneil and Perl (1984)

Objective

Complexity comparision between TOTAL DOMINATING SET and OPEN PACKING

Graph Class	TOTAL DOMINATING SET	OPEN PACKING
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Bipartite Graphs	NPC [5]	NPC [2,6]
Split Graphs	NPC [7]	NPC [3]
?	P	NPC
?	NPC	P

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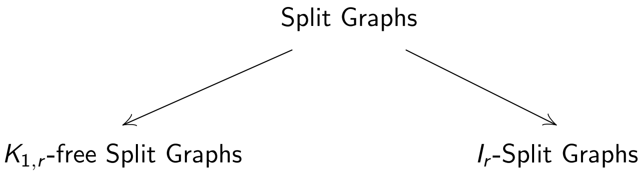
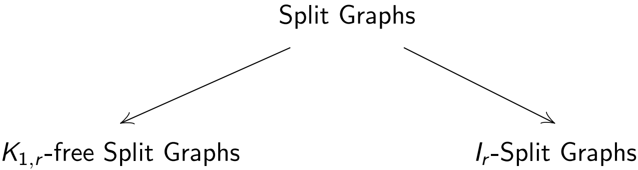
Split Graphs



Split Graphs



Split Graphs A graph G is called a split graph if there exists a partition $C \cup I$ of the vertex set such that C is a clique and I is an independent set.

Split Graphs

Split Graphs

$K_{1,r}$-free Split Graphs

I_r-Split Graphs

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$G(C \cup I, E)$ is a $K_{1,r}$-free split graph $\implies |N(v) \cap I| \leq r - 1$ for every vertex $v \in C$ [8]

[8]-Renjith and Sadagopan (2020)

Split Graphs

Split Graphs

$K_{1,r}$-free Split Graphs $\quad$ I_r-Split Graphs

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$G(C \cup I, E)$ is a $K_{1,r}$-free split graph $\Longrightarrow |N(v) \cap I| \leq r - 1$ for every vertex $v \in C$ [8]

I_r-split graphs : Given $r \geq 1$, a split graph $G(C \cup I, E)$ is said to be an I_r-split graph if $d(v) = r$ for every $v \in I$.

[8]-Renjith and Sadagopan (2020)

Split Graphs

Split Graphs

$K_{1,r}$-free Split Graphs

I_r-Split Graphs

TDS is NPC in
$K_{1,r}$-free split graphs for $r \geq 5$ [9]
and is in P for $r \leq 4$ by [8]

TDS is NPC in
I_2-split graphs by [7]

[8]-Renjith and Sadagopan (2020)
[9]-White et al. (1985)
[7]-Corneil and Perl (1984)

Dichotomy Results

1. Open Packing is NPC in $K_{1,r}$-free split graphs for $r \geq 4$ and is polynomial time solvable for $r \leq 3$.
2. Open Packing is NPC in I_r-split graphs for $r \geq 3$ and is polynomial time solvable for $r \leq 2$.

$K_{1,4}$-free Split Graphs

Theorem 11

OPEN PACKING *is NP-complete on* $K_{1,4}$*-free split graphs.*

$K_{1,4}$-free Split Graphs

Theorem 11

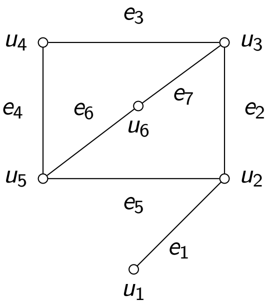
OPEN PACKING *is NP-complete on* $K_{1,4}$*-free split graphs.*

INDEPENDENT SET
Instance: A simple graph G and a positive integer $k \leq \|V(G)\|$.   *Question:* Is there an independent set of size k in G?

Karp (1972) proved that INDEPENDENT SET is NP-complete for simple graphs.

Construction 2

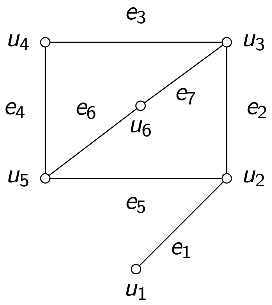
Input: A simple graph G.



G

Construction 2

Input: A simple graph G.
Output: A $K_{1,4}$-free split graph G'.



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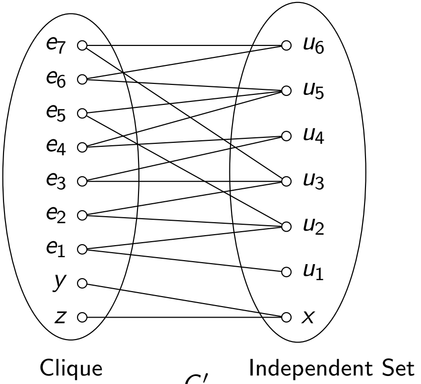
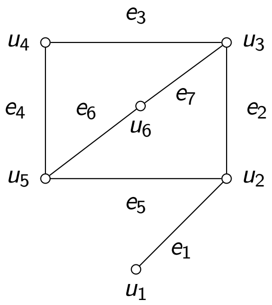


G'

Construction 2

Input: A simple graph G.
Output: A $K_{1,4}$-free split graph G'.
Gaurantee: G has an independent set of size k if and only if G' has a open packing of size $k+1$.



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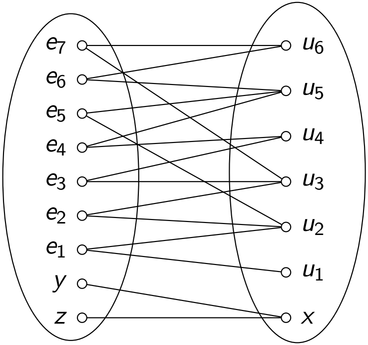
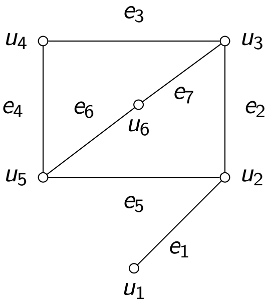


G'

Construction 2

Procedure:

Step 1 : *Replace each edge $e = uu'$ in G by a three vertex path ueu' in G'.*



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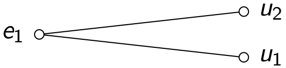

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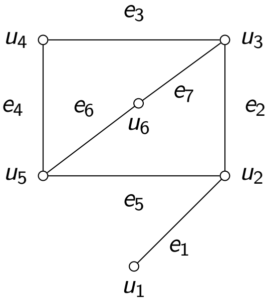

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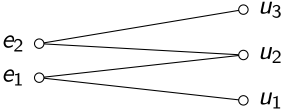

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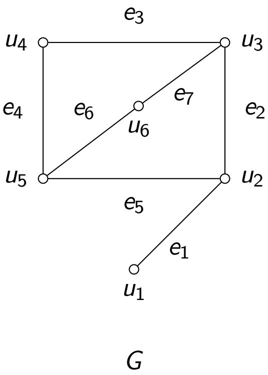
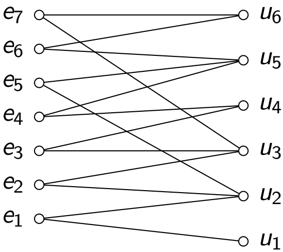


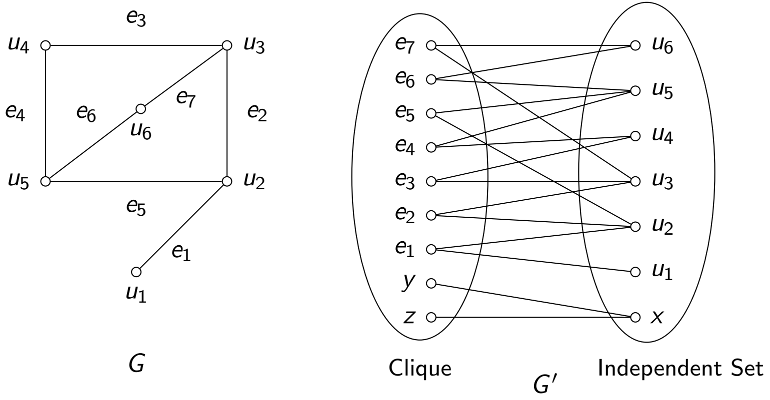


Construction 2

Procedure:

Step 1 : Replace each edge $e = uu'$ in G by a three vertex path ueu' in G'.

Step 2 : Introduce three new vertices x, y, z and two edges xy, xz in G'.



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Procedure:

Step 1 :	*Replace each edge* $e = uu'$ *in* G *by a three vertex path* ueu' *in* G'.
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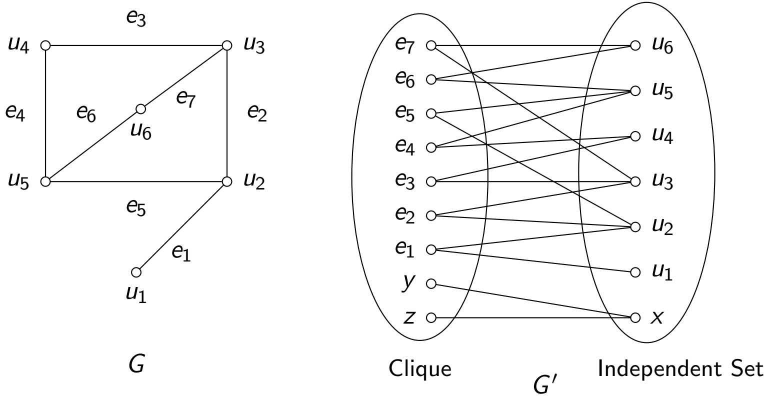
Step 3 :	*Make* $E(G) \cup \{y, z\}$ *a clique in* G'.



$K_{1,3}$-free Split Graphs

Theorem 12

OPEN PACKING *is polynomial time solvable in* $K_{1,3}$*-free split graphs.*

$K_{1,3}$-free Split Graphs

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Proof (Outline)

Let $G(C \cup I, E)$ be a $K_{1,3}$-free split graph.
Then, $|N(u) \cap I| \leq 2$ for every $u \in C$.

$K_{1,3}$-free Split Graphs

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Open Packing *is polynomial time solvable in* $K_{1,3}$*-free split graphs.*

Proof (Outline)

Let $G(C \cup I, E)$ be a $K_{1,3}$-free split graph.
Then, $|N(u) \cap I| \leq 2$ for every $u \in C$.
Case 1: $|N(u) \cap I| \leq 1$ for every $u \in C$.

Case 2: There exists $u \in C$ such that $|N(u) \cap I| = 2$.

$K_{1,3}$-free Split Graphs

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Proof (Outline)

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Then, $|N(u) \cap I| \leq 2$ for every $u \in C$.
Case 1: $|N(u) \cap I| \leq 1$ for every $u \in C$.

Then, $\rho^o(G) = \begin{cases} 2 & \text{if } |I| = 1 \text{ and } d(u) = 1 \text{ for } u \in I \\ |I| & \text{Otherwise} \end{cases}$

Case 2: There exists $u \in C$ such that $|N(u) \cap I| = 2$.

Hence, $\rho^o(G) = \begin{cases} 2 & \text{if } \exists u, v \in I \text{ such that } N(u) \cap N(v) = \emptyset \\ & \text{ or } \exists u \in I \text{ such that } d(u) = 1 \\ 1 & \text{otherwise} \end{cases}$

I_r-Split Graphs

Theorem 13

Open Packing *is NP-complete on I_r-split graphs for $r \geq 3$.*

I_r-Split Graphs

Theorem 13

OPEN PACKING *is NP-complete on I_r-split graphs for $r \geq 3$.*

For $r \geq 3$,

r-DIMENSIONAL MATCHING $\leq_P$ OPEN PACKING in I_r-Split Graphs

I_r-Split Graphs

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For $r \geq 3$,

r-DIMENSIONAL MATCHING $\leq_P$ OPEN PACKING in I_r-Split Graphs

Theorem 14

OPEN PACKING *is polynomial time solvable on I_r-split graphs for $r \leq 2$.*

For $r \leq 2$, OPEN PACKING in I_r-Split Graphs $\leq_P$ MATCHING in graphs

I_r-Split Graphs

Theorem 15

Total Dominating Set *is NP-complete on I_r-split graphs for $r \geq 2$.*

I_r-Split Graphs

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TOTAL DOMINATING SET *is NP-complete on I_r-split graphs for $r \geq 2$.*

For $r \geq 2$,

$$r\text{-HITTING SET} \leq_P \text{TOTAL DOMINATING SET in } I_r\text{-Split Graphs}$$

I_r-Split Graphs

Theorem 15

TOTAL DOMINATING SET *is NP-complete on I_r-split graphs for $r \geq 2$.*

For $r \geq 2$,

r-HITTING SET $\leq_P$ TOTAL DOMINATING SET in I_r-Split Graphs

Theorem 16

The total domination number of a I_1-split graph $G(C \cup I, E)$ is $\max\{2, |I|\}$.

Complexity

Complexity comparision between TOTAL DOMINATING SET and OPEN PACKING

Graph Class	TOTAL DOMINATING SET	OPEN PACKING
Chordal Bipartite Graphs	P [4]	P [2]
H-free Graphs	P/NP	P/NP
Bipartite Graphs	NPC [5]	NPC [2,6]
Split Graphs	NPC [7]	NPC [3]
$K_{1,4}$-free Split Graphs	P [8]	NPC
I_2-Split Graphs	NPC [7]	P

[2] - Shalu and Kirubakaran (2023)
[3]- Ramos et al. (2014)
[4] - Damaschke et al. (1990)
[5] - Pfaff et al. (1983)
[6] - Shalu et al. (2017)
[7] - Corneil and Perl (1984)
[8] - Renjith and Sadagopan (2020)

Conclusion

Future Scope

- We proved that $\gamma_t(G) \leq 2r + 2$ for a $(P_4 \cup rK_1)$-free graph.

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Future Scope

- We proved that $\gamma_t(G) \leq 2r + 2$ for a $(P_4 \cup rK_1)$-free graph. This implies that $\gamma_t(G) - \rho^o(G) \leq 2r + 1$. Can this bound be improved?
- Complexity of Total Dominating Set and Open Packing in subclasses of split graphs when regularity is imposed on

  (a) the clique set and

  (b) both the clique set and the independent set

  of the clique-independence partition of split graph.

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Thank You