On Star Partition of Split Graphs

D Divya S Vijayakumar

IIITDM KANCHEEPURAM

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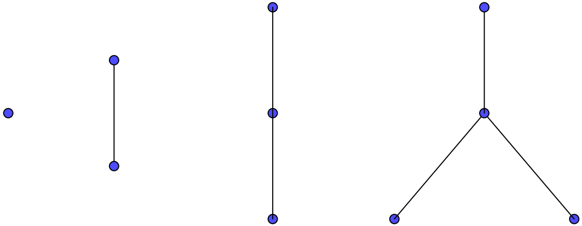

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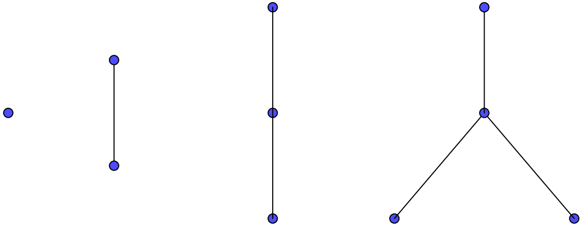


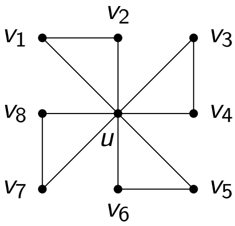
Note: Each star has a **center** vertex.

Stars in a Graph

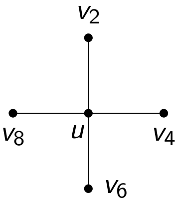
A set S of vertices from a graph G is called a **star of G** if the induced subgraph $G[S]$ is a star.

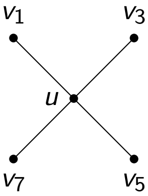
Note: A star S in a graph G partitions into an *independent set* I and and singleton set $\{x\}$: $S = \{x\} \cup I$.

Example



$G = F_4$





Definition

Star Cover

A collection of stars $\mathcal{S} = \{V_1, \ldots, V_k\}$ of a graph G is called star cover of G if $V_1 \cup \ldots \cup V_k = V(G)$.

Star Partition

A star cover $\mathcal{S} = \{V_1, \ldots, V_k\}$ of a graph G is called a *star partition* of G if the stars in it are disjoint.

$sc(G)$ and $sp(G)$

- The size of a minimum star cover of G is called the star cover number of G and is denoted $sc(G)$.
- The size of a minimum star partition of G is called the star partition number of G and is denoted $sp(G)$.

Note: The sizes of the stars do not matter!

Note: $sc(G) \leq sp(G)$.

Example

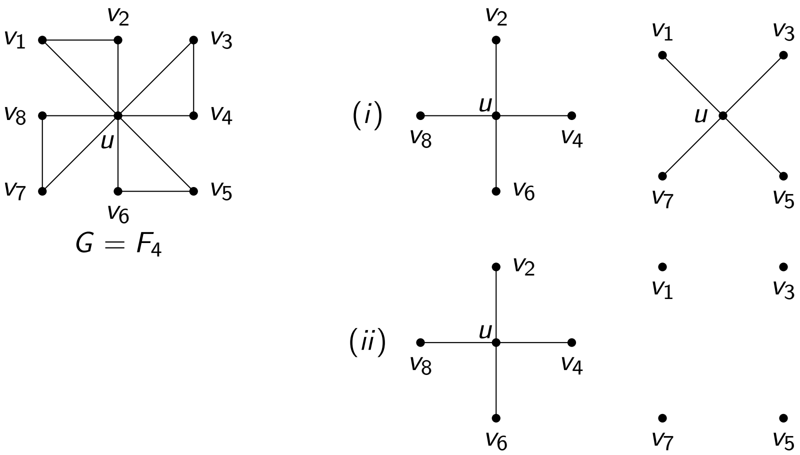


Figure: (i) $sc(G) = 2$. (ii) $sp(G) = 5$.

The Problems

Min Star Cover

Instance : A graph G.
Goal : A minimum star cover of G.

Min Star Partition

Instance : A graph G.
Goal : A minimum star partition of G.

The Decision Versions

Star Cover(D)

Instance: A graph G and a positive integer k.
Question: Does G have star cover of size at most k?

Star Partition(D)

Instance: A graph G and a positive integer k.
Question: Does G have star partition of size at most k?

A Note on Triangle-free (K_3-free) Graphs

For any triangle-free graph G,

$$sc(G) = sp(G) = \gamma(G).$$

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- Has a simple 2-approximation algorithm.
- Has linear time exact algorithms for claw-free split graphs.
- Complexity Status open for $K_{1,4}$-free split graphs.

Note

A split graph $G = (C \cup I, E)$ is $K_{1,r}$-free: Each vertex x in C has at most $r - 1$ neighbours in I.

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In This Talk: We mainly study those split graphs G for which each vertex z in I has at most a constant number of neighbours, in I: i.e., $d(z) \leq s$ for a small fixed s.

Definition

Let $G = (C \cup I, E)$ be a split graph and let r and $r_1 \leq \ldots \leq r_k$ be non-negative integers. Then:

1. G is called an r-split graph if $d(v) = r$ for each $v \in I$.

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1. G is called an r-split graph if $d(v) = r$ for each $v \in I$.
2. G is called an $(r_1, \ldots, r_k)$-split graph if $d(v)$ equals one of $r_1, \ldots, r_k$ for each $v \in I$.

STAR COVER(D) and STAR PARTITION(D) are NP-hard for

- Chordal bipartite graphs [15]
- $(C_4, C_6, \ldots, C_{2t})$-free bipartite graphs for every fixed $t \geq 2$ [7]
- Subcubic bipartite planar graphs [9, 19]
- $K_{1,5}$-free split graphs [19]
- Line graphs [5, 19]
- Co-tripartite graphs [11, 19].

Also:

- Deciding whether an input graph can be covered by *or* partitioned into three stars is NP-complete [19].
- Deciding whether an input graph can be covered by *or* partitioned into at most two stars has polynomial time algorithms.

Star Cover(D) and Star Partition(D) have polynomial time algorithms for

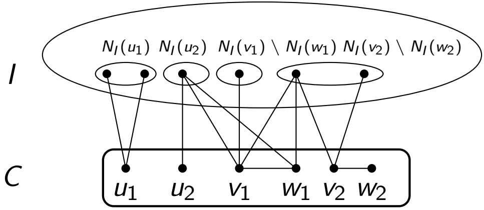
- bipartite permutation graphs [2, 8].
- convex bipartite graphs [1, 4].
- doubly-convex bipartite graphs [1].
- trees [3].
- trivially perfect graphs [12].
- co-trivially perfect graphs [12].
- claw-*free* split graphs [14].
- double-split graphs [13].

- It is NP-hard to approximate $\textsc{Star Partition}(\mathrm{D})$ within $n^{1/2-\epsilon}$ for all $\epsilon > 0$ [19, 21].
- $\textsc{Star Cover}(\mathrm{D})$ and $\textsc{Star Partition}(\mathrm{D})$ do not have any polynomial time $c \log n$-approximation algorithm for some constant $c > 0$ unless P $=$ NP [20].
- For $K_{1,r}$-free graphs
  1. $\textsc{Star Partition}(\mathrm{D})$ has a polynomial time $r/2$-approximation algorithm [10, 19].
  2. $\textsc{Star Cover}(\mathrm{D})$ has a polynomial time H_r-approximation algorithm [12]
- $\textsc{Star Cover}(\mathrm{D})$ and $\textsc{Star Partition}(\mathrm{D})$ have a polynomial time
  1. A 2-approximation algorithm for split graphs [19];
  2. $O(\log n)$-approximation algorithms for triangle-free graphs [20];
  3. $(d+1)$-approximation algorithm for triangle-free graphs of degree at most d [20].

- With solution size as the parameter, both STAR COVER(D) and STAR PARTITION(D) are
  1. $W[2]$-complete for bipartite graphs.
  2. Fixed parameter tractable for graphs of girth at least five.

- With respect to structural parameters:
  1. With vertex cover number as the parameter, the star partition problem is fixed parameter tractable.
  2. With treewidth as the parameter, the star partition is fixed parameter tractable on bounded treewidth graphs.

Structure of Stars in a Split Graph



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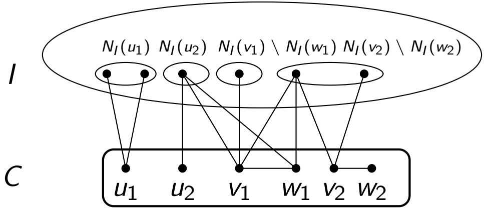

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$s = \#$ stars in star partition with one vertex from C.
$t = \#$ stars in star partition with two vertices from C.

Here $s = 2$ and $t = 2$.

Note: $s + 2t = |C|$.

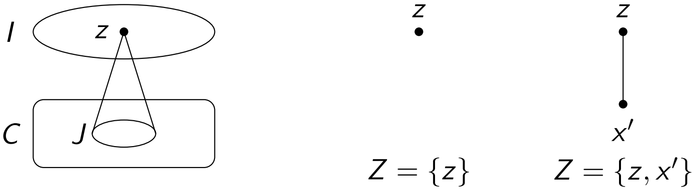
Better if the centers can **always** *be in* C*!*

Structure of Star Partitions of Split Graphs

Lemma

Let $G = (C \cup I, E)$ be a connected split graph. If G has a star partition of size k, then it also has a star partition $\mathcal{S}$ of size at most k such that each star in $\mathcal{S}$ has its center in C.

Proof



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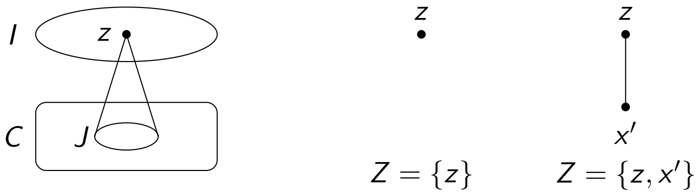

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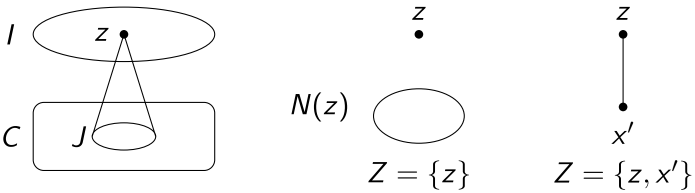

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- If $Z = \{z, x\}$, then $x \in C$ can be the center of Z.
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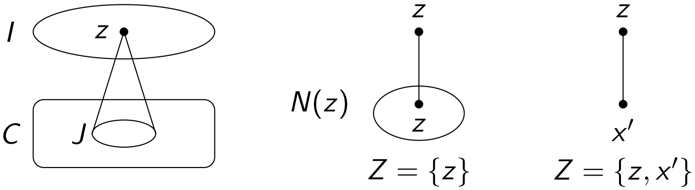

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(b) For any *ordered pair* $v, w \in C$, $X = \{v, w\} \cup [N_I(v) \setminus N_I(w)]$ is the maximal star of G with v and w as **the center–non-center pair** from C.

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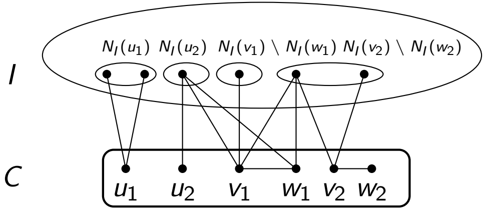
Thus, such a star partition $\mathcal{S}$ suggests a **special** three-way partition of C.

Example



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An (s,t)-partition of C

Definition

Let $G = (C \cup I, E)$ be a connected split graph with $|C| = q$. Suppose C partitions into three *ordered* sets

$$S = \{u_1, \ldots, u_s\}, T_1 = \{v_1, \ldots, v_t\}, T_2 = \{w_1, \ldots, w_t\}$$

such that

$$N_I(u_1) \cup \ldots \cup N_I(u_s) \cup [N_I(v_1) \setminus N_I(w_1)] \cup \ldots \cup [N_I(v_t) \setminus N_I(w_t)] = I.$$

Then (S, T_1, T_2) is called an (s,t)-**partition** of C.

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An (s,t)-partition of C corresponds to a star partition of of G of size $s+t$.

Results on (s, t)-partition of C

Lemma

Let $G = (C \cup I, E)$ be a connected split graph with $|C| = q$. Let s and t be non-negative integers such that $s + 2t = q$. Then G has a star partition of size $k = s + t$ if and only if C has an (s, t)-partition.

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Lemma

Let $G = (C \cup I, E)$ be a connected split graph with $|C| = q$ and $|I| = p$ and let s, t be any non-negative integers. Then we can decide whether C has an (s, t)-partition in time $O(q^{2t+1}p)$. LEX Reference.

A Characterization of Split Graphs with $sp(G) = \omega(G)$

Theorem

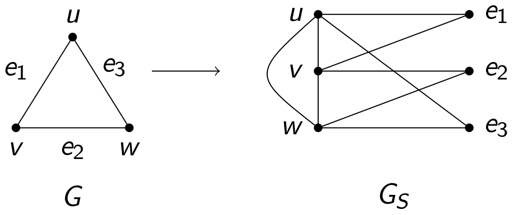
Let $G = (C \cup I, E)$ be a connected split graph with $C = \{x_1, \ldots, x_q\}$ as a maximum clique of G so that $\omega(G) = |C| = q$. Then $sp(G) = \omega(G)$ if and only if for every ordered pair (i, j) with $1 \leq i, j \leq q$ and $i \neq j$,

- *either $N_I(x_j)$ has a vertex of degree one*
- *or $N_I(x_j) \cap N_I(x_j)$ has a vertex of degree two (or both).*

The Case of 2-Split Graphs

Definition

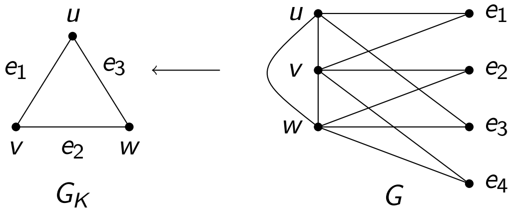
Let $G = (V, E)$ be any graph. Then the **split division** of G, denoted G_S, is the 2-split graph $G_S = (C \cup I, E_S)$ obtained from G by taking $C = V(G)$ as the clique part and $I = E(G)$ as the independent part and making each vertex $e = uv$ in $I = E(G)$ adjacent to its end vertices u and v in $C = V(G)$.



The Case of 2-Split Graphs

Definition

Let $G = (C \cup I, E)$ be a 2-split graph. Then the **kernel** of G, denoted G_K, is the graph $G_K = (V_K, E_K)$ with vertex set $V_K = C$ and edge set $E_K = \{vw \mid N_G(u) = \{v, w\} \text{ for some } u \in I\}$.

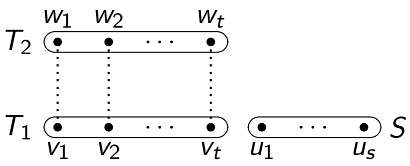


The Case of 2-Split Graphs

Lemma

Let $G = (C \cup I, E)$ be a 2-split graph and let $G_K = (V_K, E_K)$ be its kernel. Then G has $S = \{u_1, \dots, u_s\}$, $T_1 = \{v_1, \dots, v_t\}$ and $T_2 = \{w_1, \dots, w_t\}$ as an (s,t)-partition of C if and only if G_K has

- *$v_j w_j$ as a non-edge for each $1 \le j \le t$;*
- *$\{w_1, \dots, w_t\}$ as an independent set.*



(s,t)-partition of C.

Improved NP-completeness Results

Theorem

STAR PARTITION(D) *is NP-complete even when restricted to* $K_{1,5}$*-free 2-split graphs.*

The Reduction

Reduction from the following NP-complete problem.

2-3-Independent Set

Input : A graph $G = (V, E)$ with $|V| = 2\ell$ and $d(v) = 2$ or 3, for all $v \in V$, also G has a perfect matching.

Question : Does G have an independent set of size k, where $k \leq \ell - 2$.

- This NP-completeness result on independent sets follows from a simple reduction from the Max2Sat problem restricted to those instances in which
  - each clause has exactly two literals,
  - each variable occurs exactly *thrice*,
  - each literal occurs at least *once*.
- Max2Sat restricted to this instances is NP-hard [17].

The Reduction

Let (G, k) be an instance of the 2-3 independent set problem.

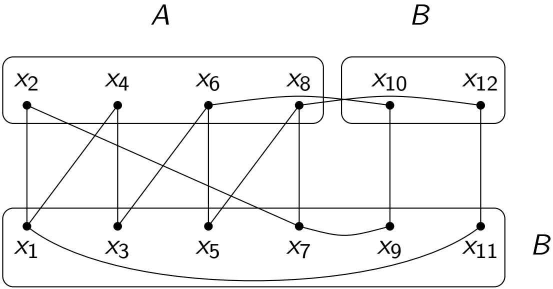
Transformed instance $(G', k`)$ with $G_s = (C \cup I, E_s)$ and $k' = 2\ell - k$.
Note: $G_s = (C \cup I, E_s)$ is $K_{1,5}$-free split graph.

Claim: *G has an independent set of size k if and only if G_s has a star partition of size $k' = 2\ell - k$.*

Suppose G has an independent set of size k, say A

.

- $k = |A| = 4$.



$G = (V, E)$

- Let $B = V(G) \setminus A$.

Suppose G has an independent set of size k, say A

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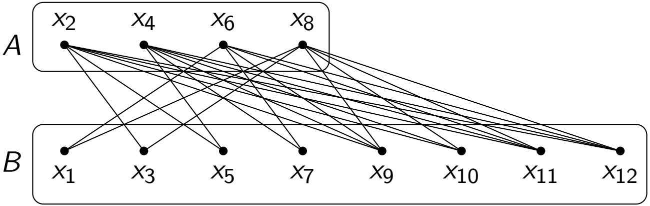
- Let $H = (A \cup B, E_H)$ be the bipartite graph with
$$E_H = \{ab : a \in A, b \in B \text{ and } ab \notin E(G)\}.$$
- H satisfies Hall's condition.
- So, H has a matching, say M, saturating A.



$H = \overline{G}[A, B]$

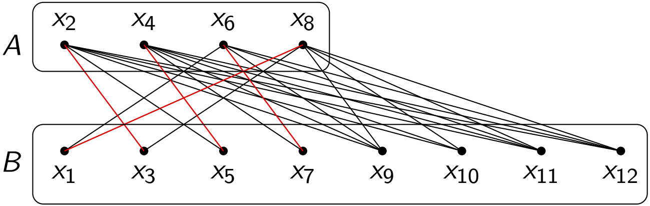
Suppose G has an independent set of size k, say A

.

- Let $H = (A \cup B, E_H)$ be the bipartite graph with

$$E_H = \{ab : a \in A, b \in B \text{ and } ab \notin E(G)\}.$$

- H satisfies Hall's condition.
- So, H has a matching, say M, saturating A.



$H = \overline{G}[A, B]$

Suppose G has an independent set of size k, say A

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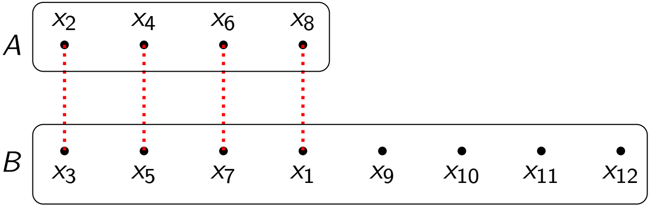

Matching of non-edges in G

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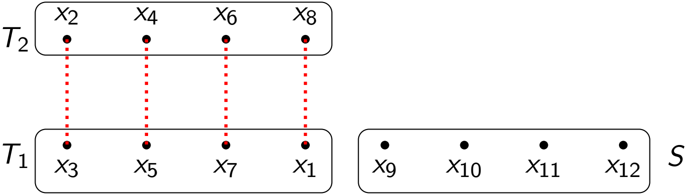

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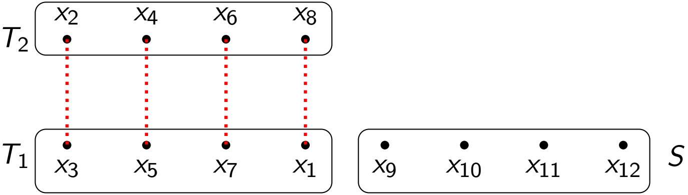


Matching of non-edges in G

- Form an $(2(\ell - k), k)$-partition of the clique part C of the 2-split graph G_S.
- Then, by Lemma, G_S has a star partition of size $2\ell - k$.

Converse

Suppose $G_S = (C \cup I, E_S)$ has a star partition of size $2\ell - k$.

Now, since $|C| = |V(G)| = 2\ell$, C has an $(2(\ell - k), \ell)$-partition, say $S = \{u_1, \ldots, u_{2(\ell-k)}\}$, $T_1 = \{v_1, \ldots, v_k\}$ and $T_2 = \{w_1, \ldots, w_k\}$.

But G_S is a 2-split graph with G as its kernel.

So, by Lemma, $\{w_1, \ldots, w_k\}$ is an independent set of G of size k.

More NP-completeness Results

Theorem

It is NP-complete to decide whether $sp(G) = \lceil \omega(G)/2 \rceil$ *even when the instances are restricted to* $K_{1,6}$*-free 2-split graphs.*

Theorem

$\textsc{Star Partition}(\mathrm{D})$ *is NP-complete even when restricted to* $(1, r)$*-split graphs for each fixed* $r \geq 2$.

Parameterized Complexity Results

We study the problems in the Parameterized Complexity Framework and consider three natural parameterizations.

Parameterization I: Solution Size as the Parameter.

STAR PARTITION(D)

Instance : A connected split graph G and a positive integer k.
Parameter : k.
Question : Does G have a star partition of size k?

Theorem

STAR PARTITION(D) *is fixed parameter tractable. In fact, it has an $O((2k)^{2k+1}n)$ time algorithm.*

Proof.

- Let $G = (C \cup I, E)$ and suppose $|C| = q$ and $|I| = p$.
- Then $q/2 \leq sp(G) \leq q$. So, we now assume that $q/2 \leq k \leq q$.
- By Lemma 4, G has a star partition of size k if and only if C has an (s, t)-partition for $(s, t) = (2k - q, q - k)$.
- Now $q \leq 2k$ and $t = q - k \leq k$.
- Lemma 5, for any non-negative integer pair (s, t), we can decide whether C has an (s, t)-partition in time $O(q^{2t+1}p)$.
- This implies that deciding whether C has an (s, t)-partition with $(s, t) = (2k - q, q - k)$ can be decided in time $O((2k)^{2k+1}n)$

□

Parameterization II: Parameterizing above a Quaranteed Value

Note: For any graph G, $sp(G) \geq \lceil \omega(G)/2 \rceil$.

Star Partition (AQ)

Instance : A graph G and a positive interger k.
Parameter : k.
Question : Is $sp(G) \leq \lceil \omega(G)/2 \rceil + k$?.

Theorem

Star Partition (AQ) *is para-NP-hard even when restricted to* either *(1)* $K_{1,6}$*-free* $(0, 2)$*-split graphs* or *(2)* $K_{1,5}$*-free* $(0, 1, 3)$*-split graphs.*

Parameterization III: Saving k Stars

Note: For any connected split graph G, $sp(G) \leq \omega(G)$.

Star Partition ($\omega - k$)

Instance : A connected split graph G and a positive integer k.
Parameter : k.
Question : Does G have a star partition of size $\omega(G) - k$?

Theorem

Star Partition ($\omega - k$) *is $W[1]$-hard even for $(1, 2)$-split graphs.*

$W[1]$-hardness: The Reduction

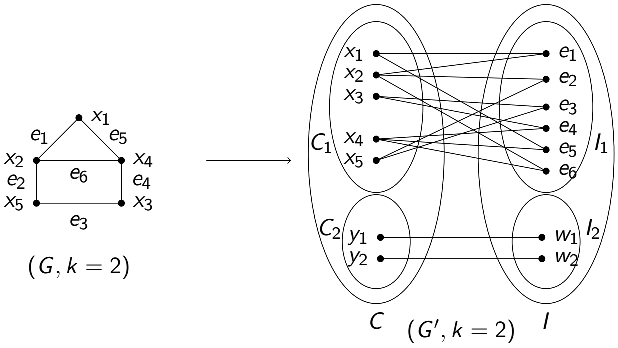
- We give a polynomial time FPT reduction from the independent set problem parameterized by the solution size k.
- The latter problem is $W[1]$-complete [6].

The Reduction

Let (G, k) be an instance of the independent set problem.

We transform this into a split graph $G' = (C \cup I, E')$, preserving the parameter k.

$$G \to G_s = (C_1 \cup I_1, E) \to G' = G_s \cup kK_2$$



- $\omega(G') = n + k$

Claim: G has an independent set of size k if and only if $sp(G') \leq \omega(G') - k$

1. G has an independent set of size k if and only if it has a vertex cover of size $n - k$.
2. G has a vertex cover of size $n - k$ if and only if, in G', $n - k$ vertices of C_1 are adjacent to all vertices in I_1.
3. The latter happens if and only if $sp(G') \leq n = \omega(G') - k$.

Theorem

STAR PARTITION $(\omega - k)$ *is in the class* $W[3]$.

Proof: We construct a circuit $\mathcal{C}$ such that it has a satisfying assignment of size k *if and only if* G has a star partition of size $\omega(G) - k$.

Or, *if and only if,* G has an (s, t)-partition with $t = k$.

Fact

Suppose (S, T_1, T_2) is an (s, t)-partition of C, where $C = \{c_1, \ldots, c_q\}$.

Without loss of generality, let:

- $S = \{c_1, \ldots, c_s\}$.
- $T_1 = \{c_{s+1}, \ldots, c_{s+t}\}$.
- $T_2 = \{c_{s+t+1}, \ldots, c_{s+2t=q}\}$.

Then $s + 1, \ldots, s + t, s + t + 1, \ldots, s + 2t = q$ are distinct.

The Construction of the Circuit

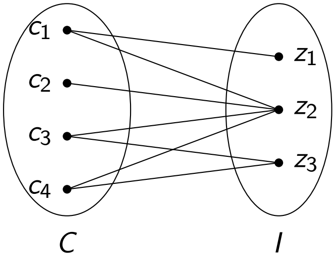
We have $C = \{c_1, \ldots, c_q\}$. Let $I = \{z_1, \ldots, z_p\}$.

Notation: $X_i = N_I(c_i)$ and $X_{ij} = X_i \setminus X_j = N_I(c_i) \setminus N_I(c_j) \quad (i \neq j)$.

The ciruit will have an input variable corresponding to each X_{ij}.

A satisfying assignment of size k pick the sets T_1 and T_2 of an (s, t)-partition for which $|T_1| = |T_2| = k$.

The Construction of the Circuit: An Example



$G = (C \cup I, E)$

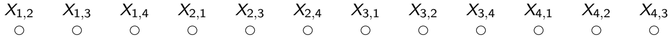
$X_1 = \{z_1, z_2\}$; $\quad X_{1,2} = X_{1,3} = X_{1,4} = \{z_1\}$;

$X_2 = \{z_2\}$; $\quad X_{2,1} = X_{2,3} = X_{2,4} = \emptyset$;

$X_3 = \{z_2, z_3\}$; $\quad X_{3,1} = X_{3,2} = \{z_3\}; X_{3,4} = \emptyset$;

$X_4 = \{z_2, z_3\}$; $\quad X_{4,1} = X_{4,2} = \{z_3\}; X_{4,3} = \emptyset$;

Circuit



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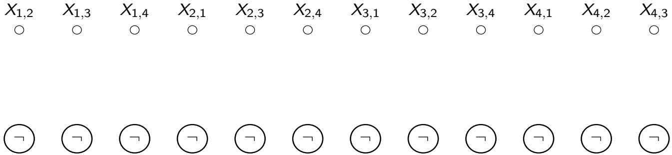

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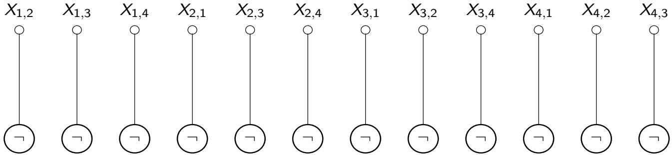

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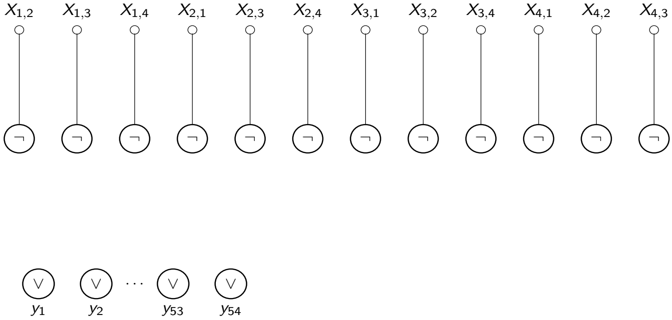

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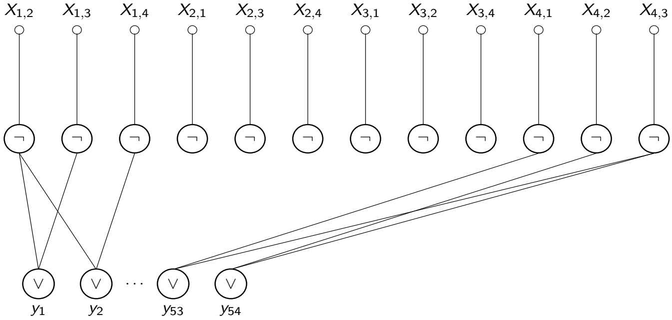

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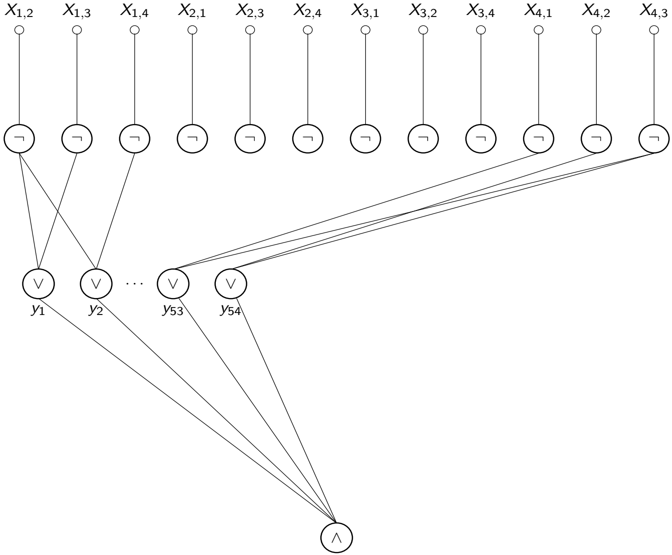

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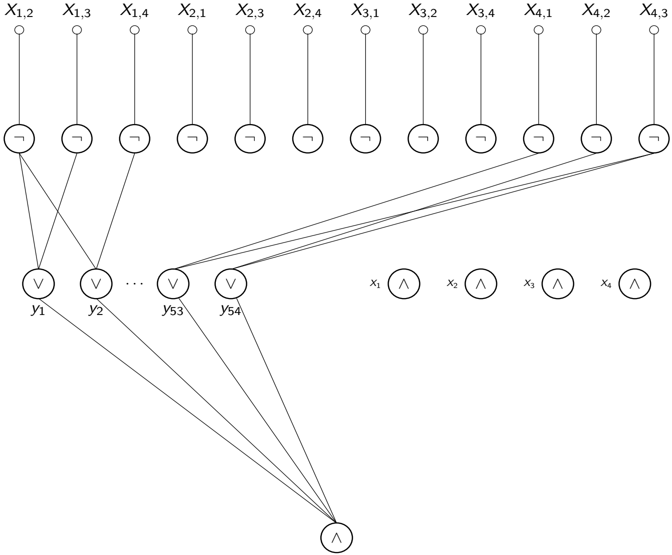

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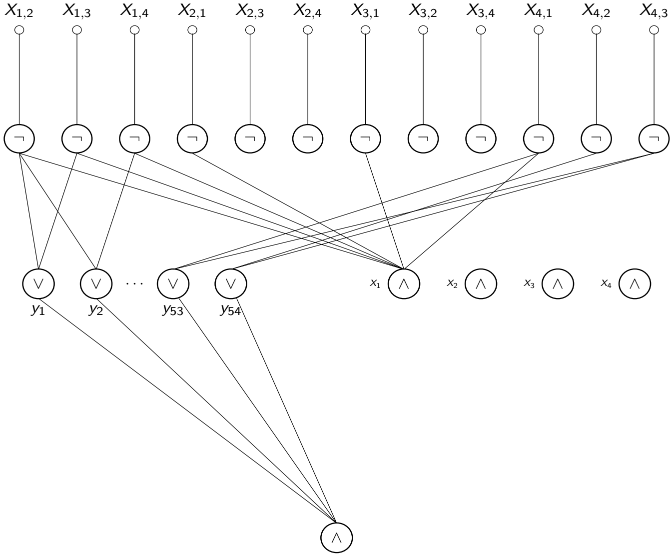
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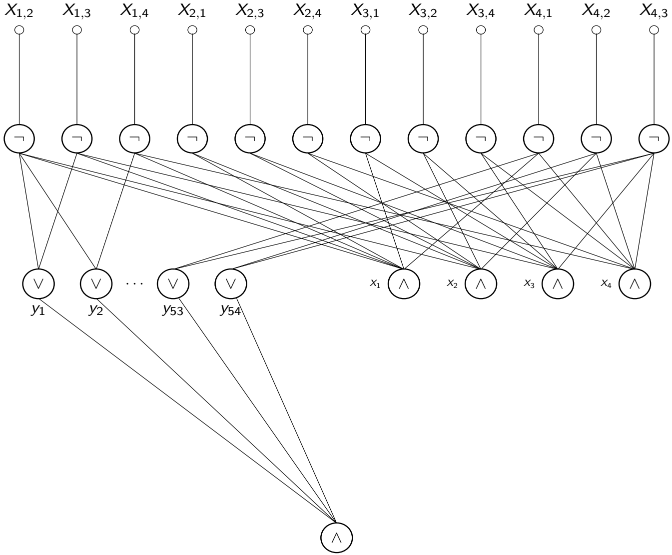

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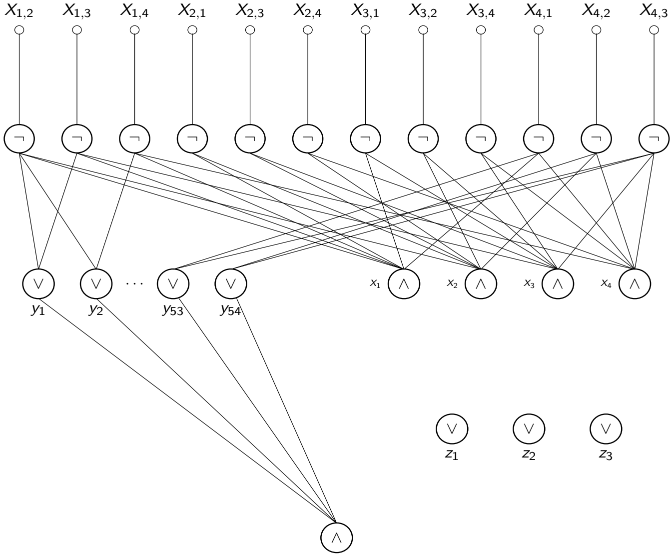

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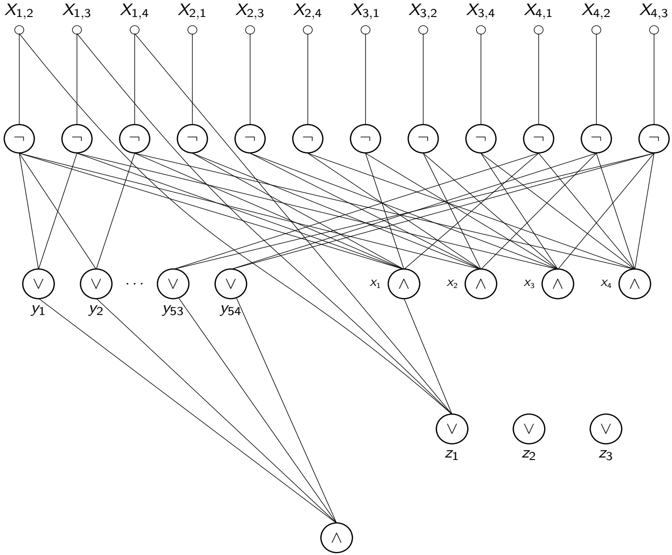

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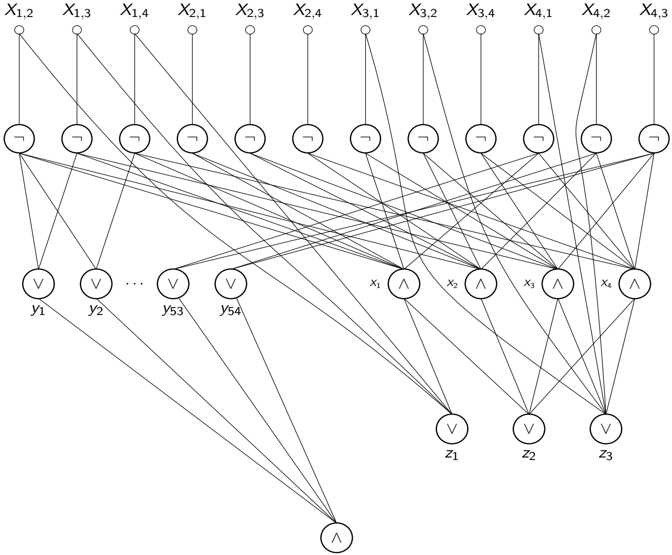
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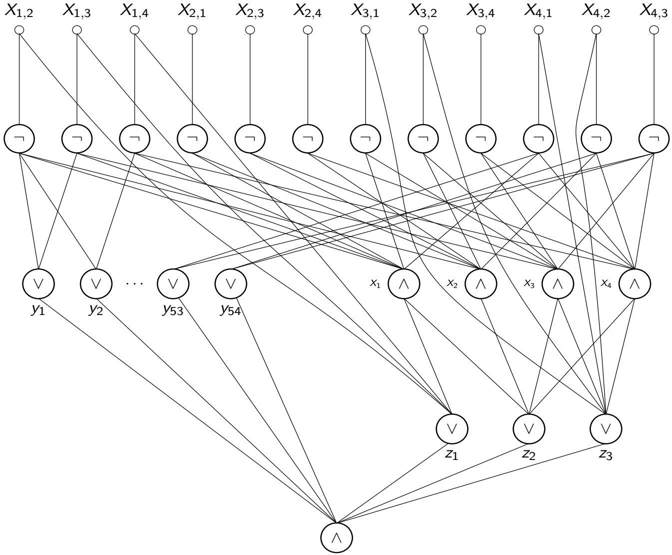

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Claim: The circuit $\mathcal{C}$ has a satisfying assignment of size k if and only if $sp(G) = \omega(G) - k$.

Suppose the circuit $\mathcal{C}$ has a satisfying assignment of size k.

- Without loss of generality, assume that the k variables $X_{s+1,s+k+1}, X_{s+2,s+k+2}, \ldots, X_{s+k,s+2k}$ are assigned the value 1.
- Then, by the design of the circuit $S = \{c_1, c_2, \ldots, c_s\}$, $T_1 = \{c_{s+1}, c_{s+2}, \ldots, c_{s+k}\}$ and $T_2 = \{c_{s+k+1}, c_{s+k+2}, \ldots, c_{s+2k=q}\}$ form an $(q - 2k, k)$ partition of C.
- So, $sp(G) = q - k = \omega(G) - k$.

Converse

Suppose $sp(G) = \omega(G) - 2k$.

- Then C has a $(q - 2k, k)$-partition.
- Without loss of generality, assume that $S = \{c_1, c_2, \ldots, c_s\}$, $T_1 = \{c_{s+1}, c_{s+2}, \ldots, c_{s+k}\}$ and $T_2 = \{c_{s+k+1}, c_{s+k+2}, \ldots, c_{s+2k=q}\}$ form a $(q - 2k, k)$ partition of C.
- Then, by the design of the circuit $\mathcal{C}$, assigning the k input nodes $X_{s+1,s+k+1}, X_{s+2,s+k+2}, \ldots, X_{s+k,s+2k}$ to 1 and the rest to 0 provides a satisfying assignment of size k.

Polynomial Time Algorithmic Results

A polynomial time 3/2-approximation algorithm for *certain* 2-split graphs.

Definition

Let $\mathcal{G}$ be any graph class. Then $\mathcal{S}(\mathcal{G})$ denotes the set of those 2-split graphs for which the kernel is in $\mathcal{G}$.

Theorem

Let $\mathcal{G}$ be any graph class for which the maximum independent set problem has a polynomial time algorithm. Then MIN STAR PARTITION *has a polynomial time 3/2-approximation algorithm for the graph class $\mathcal{S}(\mathcal{G})$.*

The Idea

Let $G = (C \cup I, E)$ be a 2-split graph with $|C| = q$.

1. Suppose an optimal star partition of G has
   1. s_o stars with one vertex from C;
   2. t_o stars with two vertices from C.
2. Then $sp(G) = s_o + t_o$ and $s_o + 2t_o = q$.
3. Also this solutions corresponds to an (s_o, t_o)-partition of C.

Now consider any (s, t)-partition of C.

We will have $s \geq s_o$ and $t \leq t_o$.

Now consider any (s, t)-partition of C.

We will have $s \geq s_o$ and $t \leq t_o$.

Let $\ell = t_o - t \geq 0$.

Then $s = s_o + 2\ell$ and $t = t_o - \ell$.

(since $s + 2t = q = s_o + 2t_o$.)

We have $s = s_o + 2\ell$ and $t = t_o - \ell$. Also

$$t_o - \ell \geq \frac{t_o}{2} \iff \ell \leq \frac{t_o}{2}$$

Suppose $s_o = 0$.

- Suppose $t = t_o - \ell \geq \dfrac{t_o}{2}$, then

$$s + t \leq \frac{3}{2} sp(G).$$

Also

$$t_o - \ell \geq \frac{t_o - 1}{2} \iff \ell \leq \frac{t_o + 1}{2}$$

Suppose $s_o \geq 1$.

- Suppose $t = t_o - \ell \geq \dfrac{t_o - 1}{2}$, then

$$s + t \leq \frac{3}{2} sp(G).$$

Algorithm: *Approximate-2-Split*

Input: A 2-split graph $G = (C \cup I, E)$.
Output: A star partition of G.

1. If $s_o = 0$, find an (s, t)-partition of C with $t \geq t_o/2$.
2. If $s_o \geq 1$, find an (s, t)-partition of C with $t \geq (t_o - 1)/2$.
3. Output a star partition of G corresponding to the (s, t)-partition of C found.

Finding a Suitable (s, t)-Partition

Let G_K be the kernel of the input 2-split graph.

Let $r = \alpha(G_K)$.

By property of (s_o, t_o)-partition:

$$s_o = 0 \implies q = 2t_o \text{ and } r \geq t_o.$$

So,

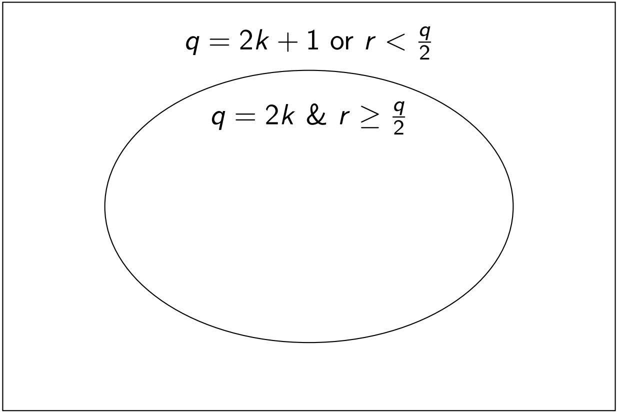
$$s_o = 0 \implies q \text{ is even } \textit{and } r \geq q/2.$$

And

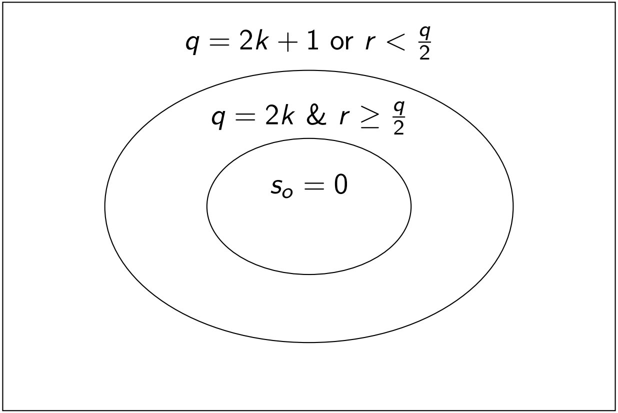
$$q \text{ is odd } \textit{or } r < q/2 \implies s_0 \geq 1.$$

Classification of 2-Split Graphs

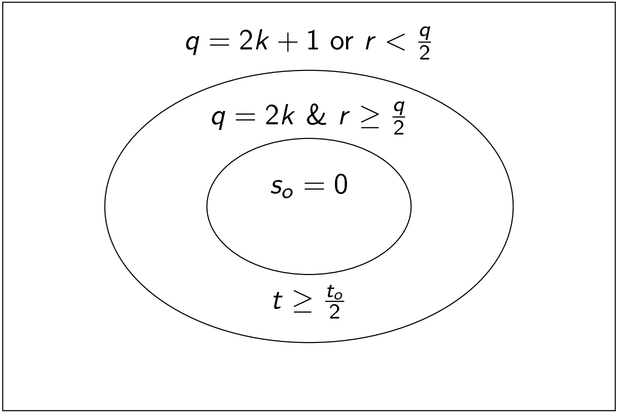
Classification of 2-Split Graphs



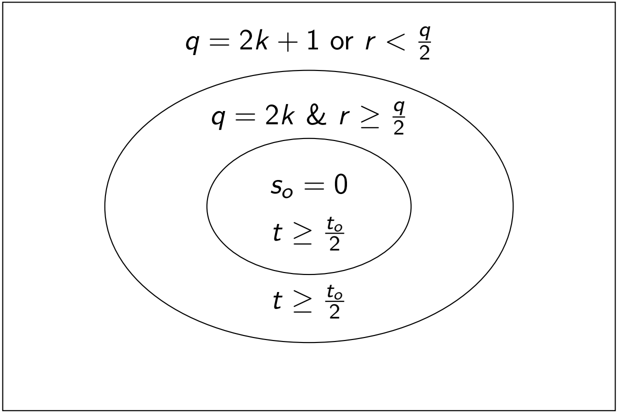
Classification of 2-Split Graphs



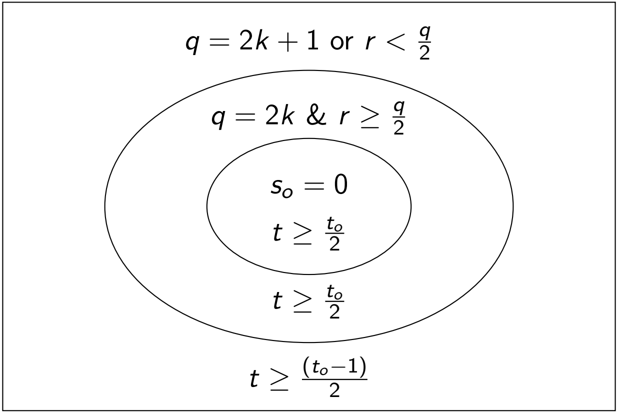
Classification of 2-Split Graphs



Classification of 2-Split Graphs



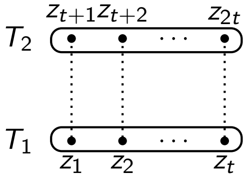
Classification of 2-Split Graphs



Idea for Suitable (s, t)-partition

Let $J = \{z_1, \ldots, z_r\}$ be a maximum independent set in the kernel G_K.

- Let $t = [r/2]$. Then $t \geq k/2 \geq t_o/2$.
- Also $T_1 = \{z_1, \ldots, z_t\}$, $T_2 = \{z_{t+1}, \ldots, z_{2t}\}$ and $S = C \setminus (T_1 \cup T_2)$ provides an (s, t)-partition of C with $t \geq t_o/2$.



Polynomial Time Algorithms

Lemma

Let G be any graph and let I be any set of vertices in G such that any pair of vertices in I are at distance three or more in G. Then $sp(G) \geq |I|$. Consequently, $sp(G) \geq \alpha(G^2)$.

Note: Computing $\alpha(G^2)$ is NP-hard even for 3-split graphs. This follows from a reduction from the NP-complete EXACT3COVER problem [9].

Corollary

Let $G = (C \cup I, E)$ be a split graph and let I_1 denote the set of all degree one vertices in the independent part I. Then $sp(G) \geq |N(I_1)|$.

Corollary

If $G = (C \cup I, E)$ is a 1-split graph, then $|N(I)| = \alpha(G^2)$.

$sp(G)$ for 1-split graph

Theorem

If G is a 1-split graph, then $sp(G) = \max(\lceil \omega(G)/2 \rceil, \alpha(G^2))$*.*
Consequently, Min Star Partition *has a linear time exact algorithm for* $(0, 1)$*-split graphs.*

Further Scope

- Determine the computational complexity of star partition on r-split graphs for each fixed $r \geq 3$.
- It would be interesting to obtain a factor $3/2$ (or better) polynomial time approximation algorithm for $\textsc{Min Star Partition}$ on at least *all* of 2-split graphs.
- Designing better than factor 2 approximation algorithms for $\textsc{Min Star Partition}$ on split graphs remains an interesting algorithmic problem.
- Does $\textsc{Star Partition}$ $(\omega - k)$ is $W[3]$-hard?
- Complexity status remains open even for $K_{1,4}$-free split graphs.

J. Bang-Jensen, J. Huang, G. MacGillivray, and A. Yeo.
Domination in convex bipartite and convex-round graphs.
Technical report, University of Southern Denmark, 1999.

A. Brandstädt and D. Kratsch.
On the restriction of some np-complete graph problems to permutation graphs.
In *Fundamentals of Computation Theory, Proc. 5th Int. Conf., Cottbus/Ger. 1985, Lecture Notes in Computer Science 199, 53-62*, pages 53–62. Springer, 1985.

E. Cockayne, S. Goodman, and S. Hedetniemi.
A linear algorithm for the domination number of a tree.
Information Processing Letters, 4:41–44, 1975.

P. Damaschke, H. Müller, and D. Kratsch.
Domination in convex and chordal bipartite graphs.
Information Processing Letters, 36:231–236, 1990.

D. Dor and M. Tarsi.

Graph decomposition is np-complete: A complete proof of holyer's conjecture.
SIAM Journal on Computing, 26:1166–1187, 1997.

R.G. Downey and M.R. Fellows.
Fundamentals of Parameterized Complexity.
Texts in Computer Science. Springer London, 2013.

O. Duginov.
Partitioning the vertex set of a bipartite graph into complete bipartite subgraphs.
Discrete Mathematics & Theoretical Computer Science, 16:203–214, 2014.

M. Farber and J.M. Keil.
Domination in permutation graphs.
Journal of Algorithms, 6:309–321, 1985.

M.R. Garey and D.S. Johnson.
Computers and Intractability: A Guide to the Theory of NP-Completeness.

W. H. Freeman, New York, 1979.

A.K. Kelmans.
Optimal packing of induced stars in a graph.
Discrete Mathematics, 173:97–127, 1997.

F. Maffray and M. Preissmann.
On the np-completeness of the k-colorability problem for triangle-free graphs.
Discrete Mathematics, 162:313–317, 1996.

J. Mondal and S. Vijayakumar.
Star covers and star partitions of cographs and butterfly-free graphs.
In *Proc. 10th International Conference on Algorithms and Discrete Applied Mathematics, CALDAM 2024.* Accepted.

J. Mondal and S. Vijayakumar.
Star covers and star partitions of double-split graphs.
Journal of Combinatorial Optimization. Accepted.

J. Mondal and S. Vijayakumar.
Star partition of certain hereditary graphs.

Manuscript.

H. Müller and A. Brandstädt.
The np-completeness of steiner tree and dominating set for chordal bipartite graphs.
Theoretical Computer Science, 53:257–265, 1987.

X.T Nguyen.
Induced star partition of graphs with respect to structural parameters.
Technical report, Charles University in Prague, 2023.

V. Raman, B. Ravikumar, and S. Srinivasa Rao.
A simplified np-complete maxsat problem.
Inf. Process. Lett., 65:1–6, 1998.

V. Raman and S. Saurabh.
Short cycles make w-hard problems hard: Fpt algorithms for w-hard problems in graphs with no short cycles.
Algorithmica, 52:203–225, 2008.

M.A. Shalu, S. Vijayakumar, T.P. Sandhya, and J. Mondal.
Induced star partition of graphs.

Discrete Applied Mathematics, 319:81–91, 2022.

V.V. Vazirani.
Approximation Algorithms.
Springer, 2013.

D. Zuckerman.
Linear degree extractors and the inapproximability of max clique and chromatic number.
Theory of Computing, 3:103–128, 2007.