The Weak-Toll Function of a Graph: Axiomatic Characterizations and First-Order Non-definability

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February 15, 2024

Toll walk, [Alcon, 2015]

- Toll walks were first introduced by Alcon (2015) as a tool to characterize dominating pairs in interval graphs.
- If we represent a toll system as a graph then one can model toll walk as the entrance fee or toll that is payed only once that is at the neighbor of the first vertex when entering a system and the toll that is payed at the neighbor of the final vertex when exits out of the system.

Toll walk function [Alcon, 2015]

Definition

- A toll walk between two different vertices w_1 and w_k of a finite connected graph G are vertices $w_1, \ldots, w_k$ that satisfy the following conditions:
  - $w_i w_{i+1} \in E(G)$ for every $i \in \{1, \ldots, k-1\}$,
  - $w_1 w_i \in E(G)$ if and only if $i = 2$,
  - $w_k w_i \in E(G)$ if and only if $i = k-1$.
- The function $T : V \times V \to 2^V$ defined as $T_G(u, v) = \{x \in V(G) : x \text{ lies on a toll walk between } u \text{ and } v\}$ is called the toll walk function on G.

Toll Walk function

- **Interval graphs and a subclass AT-free graphs (Lekshmi Kamal K. Sheela, Manoj Changat, and Iztok Peterin, CALDAM-2023)**
- **Chordal graphs, trees, asteroidal triple-free graphs, Ptolemaic graphs, and distance hereditary graphs, (Manoj Changat, Jeny Jacob, Lekshmi Kamal K. Sheela, and Iztok Peterin, arXiv:2310.20237v1, https:// doi.org/10.48550/arXiv.2310.20237)**

Weak-toll walk [Mitre C. Dourado, 2022]

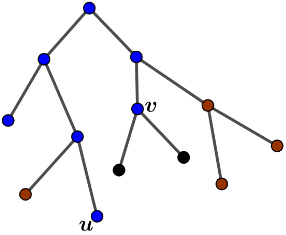
Definition

A weak-toll walk between u and v in G is a sequence of vertices of the form $W : u = w_0, w_1, \ldots, w_{k-1}, w_k = v$, where the following conditions are satisfied:

- $w_i w_{i+1} \in E(G)$ for every $i \in \{1, \ldots, k-1\}$,
- $w_0 w_i \in E(G)$ implies $w_i = w_1$,
- $w_k w_i \in E(G)$ implies $w_i = w_{k-1}$

Weak-toll function

$$W_{T_G}(u, v) = \{x \in V(G) : x \text{ lies on a weak-toll walk between } u \text{ and } v\}$$



- $W_T(u, v)$ contain both blue and red coloured vertices.
- $T(u, v)$ contain only blue coloured vertices.

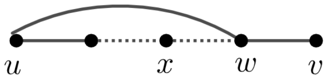
Characterization of toll and weak-toll intervals

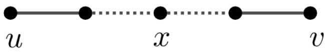
Lemma (Alcon,2015)

A vertex x is in some toll walk between two different non-adjacent vertices u and v if and only if $N[u] - \{x\}$ does not separate x from v and $N[v] - \{x\}$ does not separate x from u.

Lemma

A vertex x is in some weak-toll walk between two different non-adjacent vertices u and v if and only if there is a x, v-path that includes at most one neighbor of u and a u, x-path that includes at most one neighbor of v.

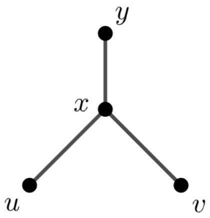


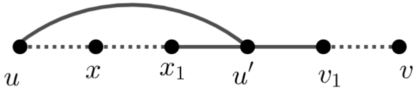


Graphs in which $T(u, v) = W_T(u, v)$

Proposition

Let G be a graph. Then $T(u, v) = W_T(u, v)$ for all u, v if and only if G is a claw-free graph.





Convex geometry with respect to W_T

Theorem (Dourado, Gutierrez, Protti, and Tondato, 2022)

The weak toll convexity of a graph G is a convex geometry if and only if G is a unit interval graph.

Transit function as a generalization of betweenness, intervals, and convexity (Mulder, 2008)

Definition

A transit function on a nonempty finite set V is a function $R : V \times V \to 2^V$ satisfying the three transit axioms

(t1) $u \in R(u, v)$, for all $u, v \in V$,

(t2) $R(u, v) = R(v, u)$, for all $u, v \in V$,

(t3) $R(u, u) = \{u\}$, for all $u \in V$.

- Betweenness : $x \in R(u, v)$
- Interval or transit sets : $R(u, v)$
- Convex set : $X \subseteq V$ is convex, if $R(u, v) \subseteq X$, for all $u, v \in X$

Transit function on Graphs

- If V is the vertex set of a graph G and R a transit function on V, then R is called a transit function on G.
- The underlying graph G_R of R is the graph (V, E), $uv \in E$ $(u \neq v)$ if and only if $R(u, v) = \{u, v\}$.
- A transit function R describes how we can move from u to v: (That is, via elements in $R(u, v)$.

- $(b1)$ if $x \in R(u, v), x \neq v$, then $v \notin R(x, u)$.
- $(b2)$ if $x \in R(u, v)$ and $y \in R(u, x)$, then $y \in R(u, v)$

Transit functions on graphs

- Interval function

$$I_G(u,v) = \{w \in V\colon\ w \text{ lies on some shortest } u,v \text{ - path in } G\} = \{w \in V : d(u,w) + d(w,v) = d(u,v)\}.$$

- Induced path transit function

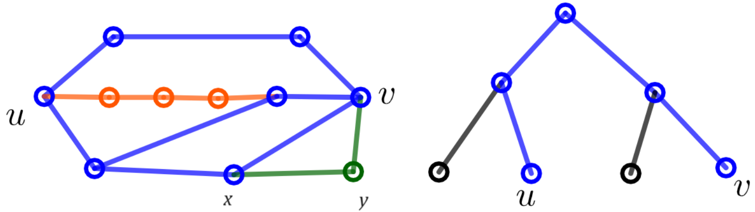
$$J_G(u,v) = \{w \in V\colon\ w \text{ lies on some induced } u,v \text{ - path in } G\}.$$

- All-paths transit function.

$$A_G(u,v) = \{w \in V\colon\ w \text{ lies on some } u,v \text{ - path in } G\}$$

$I(u,v) \subseteq J(u,v) \subseteq T(u,v) \subseteq W_T(u,v).$

Example



Motivation to axiomatic approach

- I is first-order definable on a connected graph (Nebeský [1994], Mulder and Nebeský [2009]).
- A is first-order definable on a connected graph.[C, Klavzar and Mulder, 1998]
- There does not exist a characterization of the induced path function J of a connected graph using a set of first-order axioms(Nebeský in [2002]).
- Toll walk function is not first-order definable (Manoj Changat, Jeny Jacob, Lekshmi Kamal K. Sheela, and Iztok Peterin, [2023]).

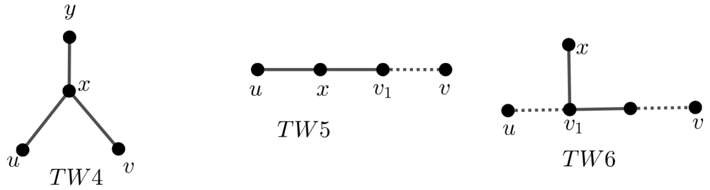
Axioms on weak-toll walks

For a transit function R on V

Axiom (TW4) $x \in R(u, v)$, $R(x, y) = \{x, y\}$, $R(y, v) \neq \{y, v\}$, $R(u, y) \neq \{u, y\}$, $\implies y \in R(u, v)$, $\forall u, v, x, y \in V$.

Axiom (TW5) $x \in R(u, v)$, $x \neq v$, $R(u, x) = \{u, x\} \implies$ there exist $v_1 \in R(x, v) \cap R(u, v)$, $v_1 \neq x$ with $R(x, v_1) = \{x, v_1\}$ and $R(u, v_1) \neq \{u, v_1\}$, $\forall u, v, x, \in V$.

Axiom (TW6) $x \in R(u, v)$, $x \neq v \implies$ there exist $v_1 \in R(x, v) \cap R(u, v)$, $v_1 \neq x$ with $R(x, v_1) = \{x, v_1\}$, $\forall u, v, x, \in V$.



Axioms on W_T

Proposition

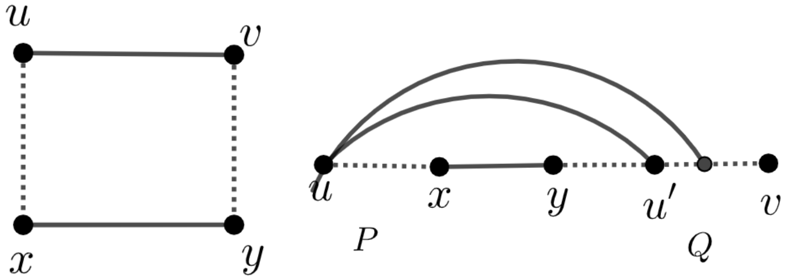
The weak-toll function satisfies the axiom (TW4), (TW5) and (TW6) on every connected graph.

Weak-toll function of chordal graphs

Axiom (JC). $x \in R(u,y)$ and $y \in R(x,v)$, $R(x,y) = \{x,y\} \implies x \in R(u,v)$, for different $u,x,y,v \in V$.

Theorem

The weak-toll function W_T of a graph G satisfies the axiom (JC) if and only if G is a chordal graph.

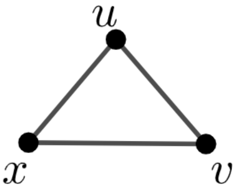


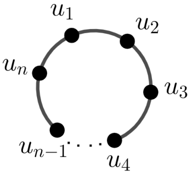
G_R is cycle free

Axiom (tr). If there exist elements $u, v, x \in V$ such that $R(u,x) = \{u,x\}$, $R(x,v) = \{x,v\}$, $u \neq v$ then $x \in R(u,v)$.

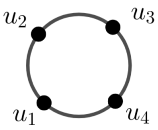
Proposition

Let R be any transit function on V. If R satisfies (JC) and (tr) then the underlying graph G_R of R is C_n-free for $n \geq 3$.





C_n



C_4

Weak-toll function of trees

Theorem

The weak-toll function W_T of a graph G satisfies the axiom (JC) and (tr) if and only if G is a tree.

Axiom on chordal graphs

Axiom (bt1). $x \in R(u,v)$, $R(u,x) = \{u,x\}$, $u \neq x \implies u \notin R(x,v)$, $\forall u,v,x$.

Proposition

The weak-toll function satisfies the axioms (bt1) on chordal graphs.

$R = W_T$

Theorem

If R is a transit function on V that satisfies the axioms (bt1), (JC), (tr), (TW4) and (TW6) then $R = W_T$ on G_R and hence G_R is connected.

Proof.

$R \subseteq W_T$

- Assume $x \in R(u, v)$
- By continuous application of the axiom (TW6), (tr) and (bt1), we obtain a sequence of vertices $v_0, v_1, \ldots, v_q$, $q \geq 2$, such that
    1. $R(v_i, v_{i+1}) = \{v_i, v_{i+1}\}$ and $v_i \neq v_{i+1}, i \in \{0, 1, \ldots, q-1\}$,
    2. $v_i \in R(u, v)$ $i \in \{0, 1, \ldots, q\}$,
    3. $v_{i+1} \in R(v_i, v)$ $i \in \{0, 1, \ldots, q-1\}$,
    4. $v_{i-1} \notin R(v_i, v)$ $i \in \{1, \ldots, q\}$.

□

proof cont..

- v_i's are distinct and this sequence needs to stop. Hence, we may assume that $v_q = v$.
- $R(u,x) = \{u,x\}$, $uxv_1 \ldots v_{q-1}v$ is a u,v-weak-toll walk.
- $R(u,x) \neq \{u,x\}$, if u is adjacent to v_m, then $uv_mv_{m-1} \ldots \; v_1xv_1 \ldots v_{q-1}v$ is a weak-toll u,v-walk.
- $x \in W_T(u,v)$. If u is not adjacent to v_i, then $u_r = u$ and $u_0u_1 \ldots u_r$ is a x,u-path in G_R and $uu_{r-1}u_{r-2} \ldots \; u_1xv_1 \ldots v_{q-1}v$ is a u,v-weak-toll walk.

$W_T \subseteq R$

Lemma

Let R be a transit function on V satisfying the axioms (JC) and (tr). If P_n, $n \geq 2$, is an induced u,v-path in G_R, then $V(P_n) \subseteq R(u,v)$.

Characterization of toll function of trees

Theorem

A transit function R on V satisfies the axioms (bt1), (tr), (J2), (JC), (TW4) and (TW6) if and only if G_R is a tree and $R = W_T$ on G_R.

Weak-toll function of (claw, hole, house, net, S_3)-free

Theorem

The weak-toll function W_T of a graph G satisfies the axiom (b1) if and only if G is (claw, hole, house, net, S_3)-free graph.

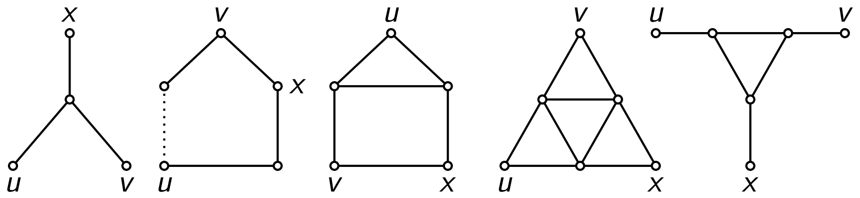


Figure: Graphs claw, hole-H, house, S_3 and net (from left to right).

Weak-toll function of unit interval graphs

Theorem

The weak-toll function W_T of a graph G satisfies the axioms (b1) and (JC) if and only if G is a unit interval graph.

Theorem

The weak-toll function W_T of a graph G satisfies the axioms (b1) and (J0) if and only if G is a unit interval graph.

Characterization of unit interval graphs

Theorem

A transit function R on V satisfies the axioms (b1), (b2) (J2), (J0), (TW4), (TW5) and (TW6) if and only if G_R is a connected unit interval graph and $R = W_T$ on G_R.

Lemma

If a transit function R on a finite non-empty set V satisfies axioms (b1) and (b2) then R satisfies axiom (TW6).

Characterization of unit interval graphs

Theorem

A transit function R on a finite non-empty set V satisfies the axioms (b1), (b2) (J2), (J0), (TW4) and (TW5) if and only if G_R is a connected unit interval graph and $R = W_T$ on G_R.

We obtain a sequence of vertices $v_0, v_1, \ldots, v_q$, $q \geq 2$, such that

1. $R(v_i, v_{i+1}) = \{v_i, v_{i+1}\}, i \in \{0, 1, \ldots, q-1\}$,
2. $R(v_{i+1}, v) \subset R(v_i, v), i \in \{0, 1, \ldots, q-1\}$,
3. $R(u, v_i) \neq \{u, v_i\}, i \in \{1, \ldots, q\}$.

we can symmetrically build a sequence $u_0, u_1, \ldots, u_r$, where $u_0 = x$, $u_r = u$ and $u_0 u_1 \ldots u_r$ is a x, u-path in G_R and hence, $u u_{r-1} u_{r-2} \ldots u_1 x v_1 \ldots v_{q-1} v$ is a u, v-weak-toll walk

Examples

Example ((J2), (b2), (J0), (TW4), (TW5) but not (b1))

Define R as : $R(u, v) = R(x, v) = R(u, x) = V$, $R(a, b) = \{a, b\}$ for all other $a, b \in V$. Then R satisfies the axioms (J2), (b2), (J0), (TW4), (TW5) and (TW6). Furthermore, $x \in R(u, v), x \neq v$ and $v \in R(u, x)$ and R do not satisfy axiom (b1).

Example ((b1), (J2), (b2), (tr), (TW4), (TW5), (TW6) but not (J0), (JC))

Define R as : $R(u, y) = \{u, x, v, y\}$, $R(x, v) = \{x, y, u, v\}$, $R(a, b) = \{a, b\}$ for all other $a, b \in V$. Then R satisfies axioms (b1), (b2), (tr), (J2), (TW4), (TW5) and (TW6). Furthermore, $x \in R(u, y)$, $y \in R(x, v)$ and $x \notin R(u, v)$ and hence R does not satisfy the axioms (J0) and (JC).

Examples

Example ((b1), (b2), (J0), (TW4), (TW5), (TW6) but not (J2), (tr))

Define R as : $R(u, v) = \{u, x, v\}$, $R(a, b) = \{a, b\}$ for all other $a, b \in V$. Then R satisfies axioms (b1), (b2), (J0), (TW4), (TW5) and (TW6). In addition $R(u, y) = \{u, y\}$, $R(y, v) = \{y, v\}$, $R(u, v) \neq \{u, v\}$ but $y \notin R(u, v)$ so that R does not satisfy the axioms (J2) and (tr).

Example ((b1), (b2), (J2), (J0), (tr), (TW5) and (TW6), but not (TW4))

Define R as : $R(u, v) = \{u, y, v\}$, $R(u, x) = \{u, y, x\}$, $R(x, v) = \{x, y, v\}$, $R(a, b) = \{a, b\}$ for all other $a, b \in V$. Then R satisfies the axioms (b1), (b2), (tr), (J2), (J0), (TW5) and (TW6). Furthermore, $y \in R(u, v)$, $R(x, y) = \{x, y\}$, $R(u, x) \neq \{u, x\}$, $R(v, x) \neq \{v, x\}$

Examples

Example ((b1), (b2), (J2), (J0), (TW4) and (TW6), but not (TW5))

Define R as: $R(u,v) = \{u,x,y,v\}$, $R(x,v) = \{x,y,v\}$, $R(a,b) = \{a,b\}$ for all the other $a,b \in V$. Then, R satisfies axioms (b1), (b2), (J2), (J0), (TW4) and (TW6). In addition to $x \in R(u,v)$, $R(u,x) = \{u,x\}$, there does not exist v_1 such that $v_1 \in R(x,v) \cap R(u,v)$, $v_1 \neq x$ with $R(x,v_1) = \{x,v_1\}$, $R(u,v_1) \neq \{u,v_1\}$ so R does not satisfy axiom (TW5).

Example ((JC), (TW4), (tr) and (TW6), but not (bt1 and (b1)).)

Define R as $R(a,b) = \{a,b\}$ for all $a,b \in V$. Then R satisfies axioms (JC), (TW4), (tr) and (TW6). But $x \in R(u,v)$, $R(u,x) = \{u,x\}$ and $u \in R(v,x)$ and R do not satisfy axiom (bt1).

Structure

Definition

The tuple $\mathbf{X} = (X, \sigma)$ is called a *structure* when X is a nonempty set called *universe*, and σ is a finite set of function symbols, relation symbols, and constant symbols called *signature or vocabulary*.

- Here, we assume that the signature contains only relation symbols.

Partial isomorphism

A map q is said to be a *partial isomorphism* from **X** to **Y** if and only if

- $dom(q) \subset X$,
- $rg(q) \subset Y$,
- q is injective and for any n-ary relation R in the signature
- $a_0, \ldots, a_l \in dom(q)$, $R^{\mathcal{X}}(a_0, \ldots, a_l)$ if and only if $R^{\mathcal{Y}}(q(a_0), \ldots, q(a_l))$.

r-move Ehrenfeucht-Fraïssé game

- The *r-move Ehrenfeucht-Fraïssé game on* **X** and **Y** is played between two players, called the *Spoiler* and the *Duplicator*.
- Each run of the game has r moves. In each move, the Spoiler plays first and picks an element from the universe X of **X** or from the universe Y of **Y**.
- The Duplicator then responds by picking an element of the other structure.
- The *Duplicator wins the run* $(a_1, b_1), \ldots, (a_r, b_r)$ if the mapping $a_i \mapsto b_i, i = 1, \ldots, r$ is a partial isomorphism from **X** to **Y**.
- The *Duplicator wins the r-move Ehrenfeucht-Fraïssé game on* **X** and **Y** if the Duplicator can win every run of the game.

[L. Libkin, 2013]

Theorem

The following statements are equivalent for two structures **X** *and* **Y** *in a relational vocabulary.*

1. **X** *and* **Y** *satisfy the same sentence* σ *with* $qr(\sigma) \leq n$.
2. *The Duplicator has an n-round winning strategy in the EF game on* **X** *and* **Y**.

Theorem

A property P *is expressible in first order logic if and only if there exists a number k such that for every two structures* **X** *and* **Y**, *if* **X** $\in \mathrm{P}$ *and Duplicator has a k-round winning strategy on* **X** *and* **Y** *then* **Y** $\in \mathrm{P}$.

Scant property, [L. Nebeský, 2002]

(SP): If $R(x, y) \neq \{x, y\} \implies R(x, y) = V$, for any $x, y \in V$.

- Ternary structure-(X, D),
- $D(x, u, y)$ - $F(x, y) = \{u \in X : D(x, u, y)\}$.
- Underlying graph of (X, D): vertex set-X, distinct vertices u and v of G are adjacent if and only if $\{x \in X : D(u, x, v)\} \cup \{x \in X : D(v, x, u)\} = \{u, v\}$.
- We call a ternary structure (X, D), 'the W'- *structure* of a graph G, if X is the vertex set of G and D is the ternary relation corresponding to W_T (that is, $(x, y, z) \in D$ if and only if y lies in some x, z- weak-toll walk).

W'-structure is Scant

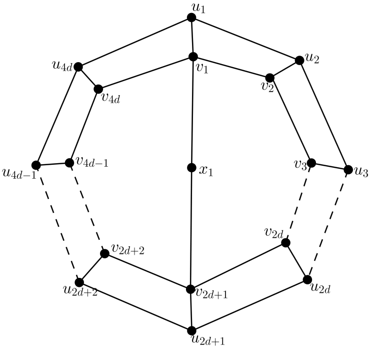


Figure: Graph H_d.

W'-structure is not Scant

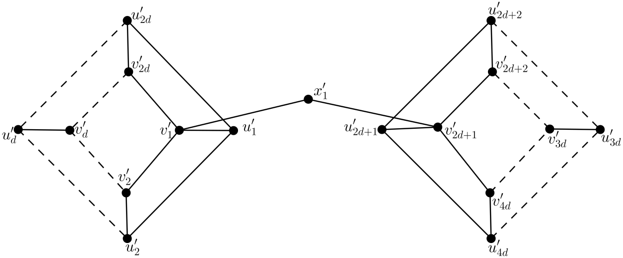


Figure: Graph H'_d.

W_T is not FO definable

Lemma

Let $d \geq 2$.

i. The W'-structure of H_d is scant.

ii. The W'-structure of H'_d is not scant.

Lemma

Let $n \geq 1$ and $d > 2^{n+1}$. If (X_1, D_1) and (X_2, D_2) are scant ternary structures such that the underlying graph of (X_1, D_1) is H_d and the underlying graph of (X_2, D_2) is H'_d, then (X_1, D_1) and (X_2, D_2) satisfy the same sentence ψ with $qr(\psi) \leq n$.

W_T is not FO definable

Theorem

There exists no sentence σ of the first-order logic of vocabulary $\{D\}$ such that a connected ternary structure is a W'-structure if and only if it satisfies σ.

Problem

Problem: Is there a first-order axiomatic characterization of the weak-toll function W_T of Ptolemaic graphs and perfect graphs?

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THANK YOU...