Unique Least Common Ancestors and Clusters in Directed Acyclic Graphs

MANOJ CHANGAT

Department of Futures Studies
University of Kerala

At

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Joined Work With

Ameera V Shanavas, Marc Hellmuth and Peter F Stadler

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Outline

DAGs and Phylogenetics

- **Directed Acyclic graph (DAG)**

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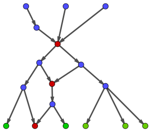
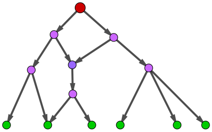
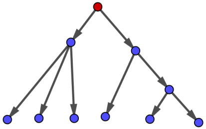


DAG with hybrid vertices



Network



Phylogenetic tree

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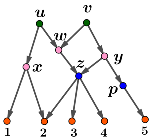

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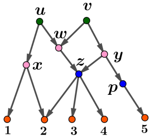

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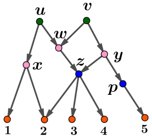

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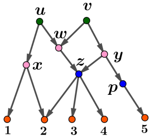

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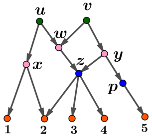

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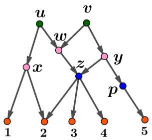

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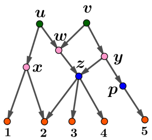

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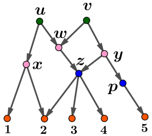

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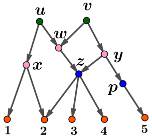

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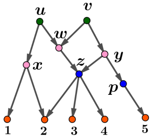

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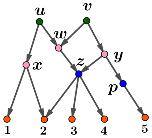

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- In a network, $\mathrm{LCA}(Y) \neq \emptyset$.

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Transit functions and Cluster systems on DAGs

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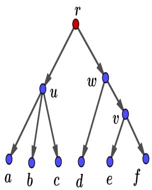
Example (3-ary transit function)



Transit functions and Cluster systems on DAGs

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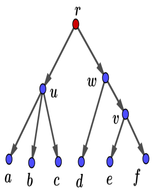

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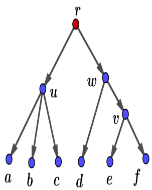

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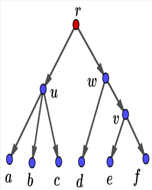

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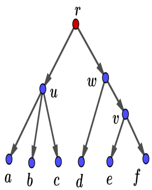

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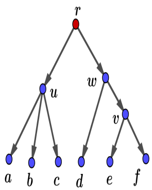

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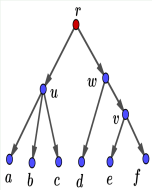

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($\mathscr{C}$ is closed under pairwise intersections)
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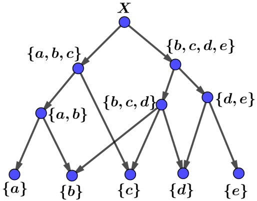

$$\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{b, c, d\}, \{d, e\}, \{a, b, c\}, \{b, c, d, e\}, X\}$$

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Standard Clustering Systems

- **Hierarchy**

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- Weak hierarchy = 2-weak hierarchy
  $\implies$ k-weak hierarchy
  $\implies$ $(k+1)$-weak hierarchy for all $k \geq 3$.

Correspondence between Hierarchy and Rooted Phylogenetic tree

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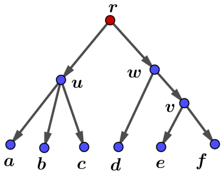

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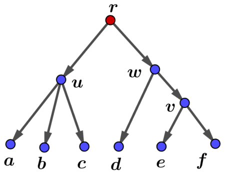

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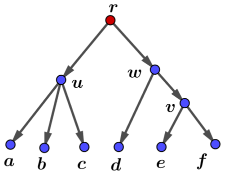

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- $\mathscr{C}$ is Hierarchy on X
  $\implies$ Hasse Diagram $\mathscr{H}(\mathscr{C})$ is a rooted phylogenetic tree with leaf set X

Observation 1 (Bertrand and Diatta, 2017)

A set system $\mathscr{C}$ is a k-weak hierarchy if and only if for every $A \in 2^X$ with $|A| > k$ there is $z \in A$ such that $z \in \text{cl}(A \setminus \{z\})$.

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A set system $\mathscr{C}$ on X is a k-weak hierarchy if and only if for every $\emptyset \neq A \subseteq X$ there exists $U \subseteq A$ with $|U| \leq k$ such that $\mathrm{cl}(A) = \mathrm{cl}(U)$.

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Lemma 2

Let R be a k-ary transit function. Then $G = \mathfrak{H}(\mathscr{C}_R)$ is a network if and only if R satisfies **(a')** for k.

Hellmuth, Schaller, Stadler, 2022

Path-Cluster-Comparability **(PCC)** Property

u and v are $\preceq_G$-comparable if and only if $\mathtt{C}(u) \subseteq \mathtt{C}(v)$ or $\mathtt{C}(v) \subseteq \mathtt{C}(u)$ for all $u, v \in V$

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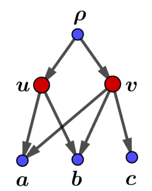
Example (G without PCC)



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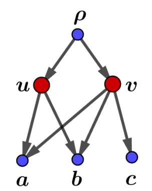

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- **Hasse diagram G of a clustering system $\mathscr{C}$ satisfies (PCC) and $\mathscr{C}_G = \mathscr{C}$**

DAGs with lca- and k-lca- property

Hellmuth, Schaller, Stadler, 2022

lca-Property

A DAG with leaf set X has the **lca-property** if $\text{lca}(A)$ is defined for all non-empty $A \subseteq X$.

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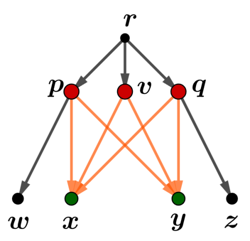
Lemma 3

If a DAG G has the lca-property, then its clustering system $\mathscr{C}_G$ is closed.

Example ($\mathscr{C}_G$ is closed $\not\Rightarrow$ G is lca network)



$\mathscr{C}_G = \{\{w\}, \{x\}, \{y\}, \{x, y\}, \{w, x, y\}, \{x, y, z\}, \{w, x, y, z\}\}$

$\mathscr{C}_G$ is closed. However, G does not have the lca-property since

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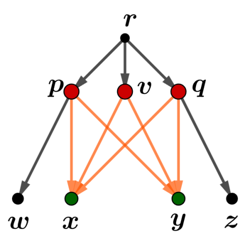

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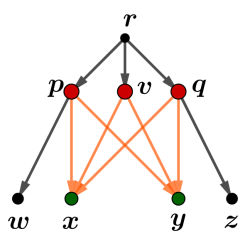

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If G is a DAG with leaf set X and the lca-property, then $\mathtt{C}(\mathrm{lca}(Y)) = \mathrm{cl}(Y)$ for all $\emptyset \neq Y \subseteq X$.

Definition

A DAG G with leaf set X has the **k-lca-property** if $\mathrm{lca}(A)$ is defined for all $A \in X^{(k)}$.

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- We define the k-ary map $R_G : X^k \to 2^X$ by

$$R_G(u_1, \ldots, u_k) := \mathtt{C}(\operatorname{lca}(u_1, \ldots, u_k)) \text{ for all } U \in X^{(k)}$$

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Proposition 2

Let G be a DAG with k-lca-property. Then R_G is a monotone k-ary transit function that satisfies $R_G = R_{\mathscr{C}_G}$. Moreover, $\mathscr{C}_G$ is pre-k-ary.

Proof

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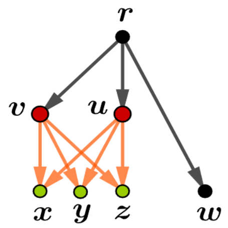
- $U \in X^{(k)}$, $\operatorname{lca}(U)$ is defined
- $(1) \implies cl(U) \in \mathscr{C}_G$

  $\implies \mathscr{C}_G$ satisfies **(KC)**

  $\implies \mathscr{C}_G$ is pre-k-ary

Example ($\mathscr{C}_G$ is pre-k-ary $\not\Rightarrow$ G is k-lca (k=2,3))



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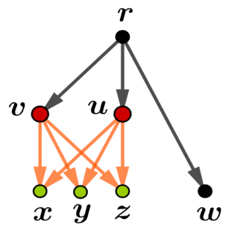

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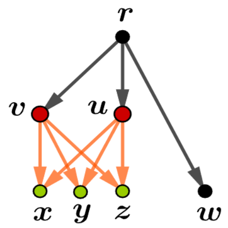

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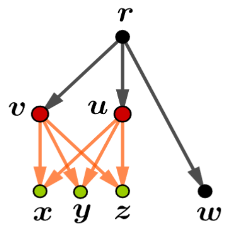

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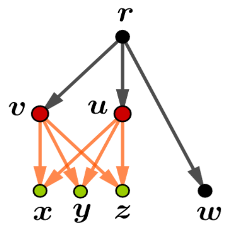

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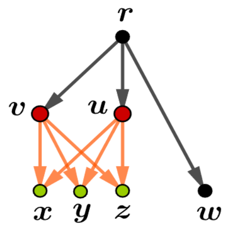

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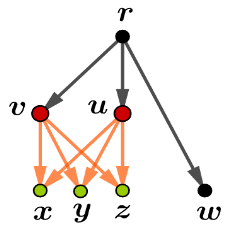

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Proposition 3

Let G be a DAG that satisfies **(PCC)**. Then, G satisfies the k-lca-property if and only if $\mathscr{C}_G$ is pre-k-ary.

Proof.

□

Proof.

- 'If part' by Prop. 2

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- Suppose G: (PCC) & $\mathscr{C}_G$: pre-k-ary
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A clustering system $\mathscr{C}$ is pre-k-ary if and only if there is a DAG G with $\mathscr{C} = \mathscr{C}_G$ and k-lca-property.

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Theorem 2

Let R be a k-ary transit function. Then R is monotone if and only if there is a DAG G with k-lca-property, which satisfies $\mathscr{C}_G = \mathscr{C}_R$ and $R = R_{\mathscr{C}_G}$.

DAGs with Strict and Strong k-lca-Property

Cluster-lca Property (Hellmuth, Schaller, Stadler, 2022)

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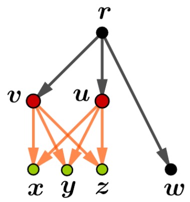

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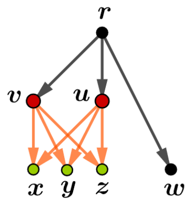

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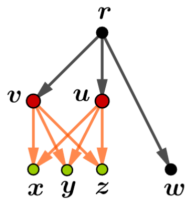

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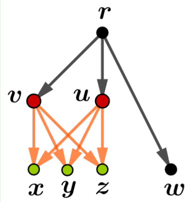

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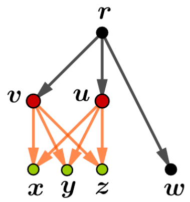

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- lca-property $\implies$ **(CL)**

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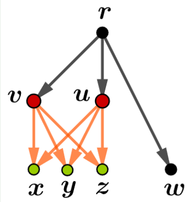

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- lca-property $\implies$ **(CL)**

Observation 4

Let G be a DAG satisfying **(CL)**. Then $\mathtt{C}(\mathrm{lca}(\mathtt{C}(v))) = \mathtt{C}(v)$ for all $v \in V(G)$.

Correspondence between Strict k-lca-property and k-ary $\mathscr{T}$-system

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Strict k-lca-Property

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Definition

Let G be a DAG with leaf set X and k-lca property. Then, G has the **strict k-lca-property** if G satisfies **(CL)**, and for every $w \in V(G)$, there is $U \in X^{(k)}$ such that $\mathrm{lca}(\mathtt{C}(w)) = \mathrm{lca}(U)$.

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Proposition 4

Let G be a DAG with leaf set X and k-lca-property. Then G has the strict k-lca-property if and only if $\mathscr{C}_G$ is a k-ary $\mathscr{T}$-system. In this case, $\mathscr{C}_G$ is identified by R_G.

Hellmuth, Schaller, Stadler, 2022

Strong lca-**property**

Definition

Let G be DAG with leaf set X and lca-property. Then, G has the **strong** lca-**property** if, for every non-empty subset $A \subseteq X$, there exists $x, y \in A$ such that $\text{lca}(A) = \text{lca}(x, y)$.

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- **G is a strong lca-network if and only if G has the lca-property and $\mathscr{C}_G$ is a weak hierarchy.**

Strong k-lca-**property**

Generalizing the **strong** lca-**property**

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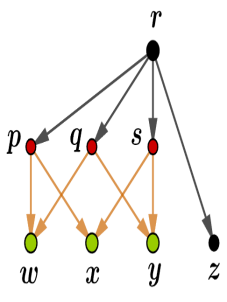
Lemma 5

If a DAG G has the strong k-lca-property, then it has the strict k-lca-property.

Example (lca network which is not Strong *k*-lca)



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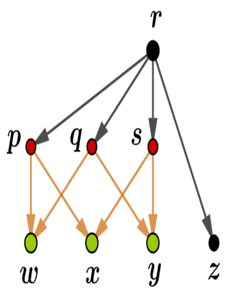

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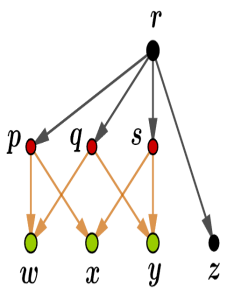

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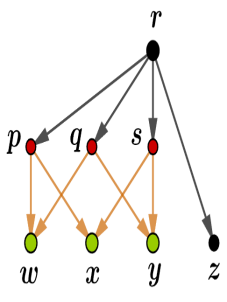

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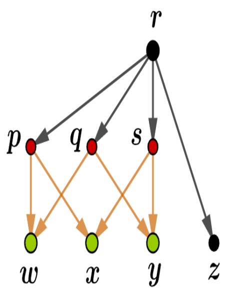

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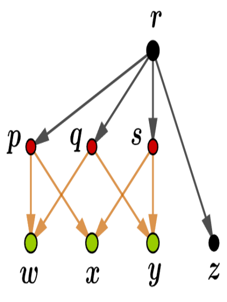

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Let G be a DAG with leaf set X and the lca-property. Then G has the strong k-lca-property if and only if $\mathscr{C}_G$ holds the following condition.
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- Let $A \subseteq X$ be non-empty
- Obs. 2(iii) & Obs. 3 $\implies$

$$\mathrm{lca}(A) = \mathrm{lca}(\mathtt{C}(\mathrm{lca}(A))) = \mathrm{lca}(\mathrm{cl}(A)) = \mathrm{lca}(\mathrm{cl}(U)) = \mathrm{lca}(\mathtt{C}(\mathrm{lca}(U))) = \mathrm{lca}(U)$$

Correspondence between Strong k-lca-property and k-weak hierarchy

Theorem 3

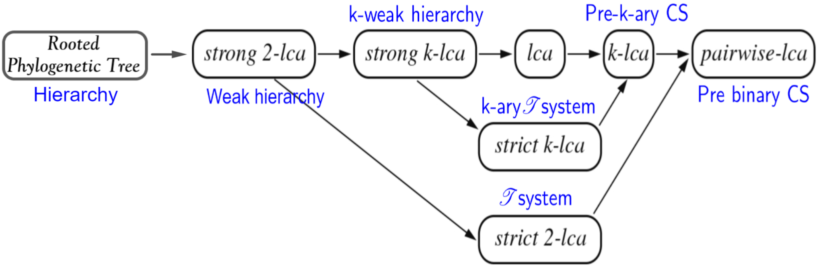
G is a DAG with the strong k-lca-property if and only if G has the lca-property and $\mathscr{C}_G$ is a k-weak hierarchy.

Proof.

- Suppose G has the strong k-lca-property
- G has the lca-property
- Prop. 5 $\implies$ (U, A, X)
- Prop. 1 $\implies$ $\mathscr{C}_G$ is a k-weak hierarchy

- Let $\mathscr{C}_G$ be a k-weak hierarchy
- By Prop. 1 $\implies$ Condition (U, A, X)
- Prop. 5 $\implies$ G has the strong k-lca-property

□



Rooted Phylogenetic Tree
Hierarchy
strong 2-lca
Weak hierarchy
k-weak hierarchy
strong k-lca
lca
Pre-k-ary CS
k-lca
pairwise-lca
Pre binary CS
k-ary 𝒯 system
strict k-lca
𝒯 system
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THANK YOU!