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Impact of Diameter and Convex Ordering for Hamiltonicity and Domination

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Roadmap of the Talk

- Introduction
- Literature Survey
- Our results
- Conclusions & Open problems

Introduction

Introduction

The Hamiltonian cycle problem asks for the presence of a cycle that visits each node exactly once.

More formally,
Instance: A graph G
Question: Does there exist a cycle in G that visits each node exactly once?

Similarly, the Hamiltonian path problem asks for the presence of a path that visits each node exactly once.

Applications

Hamiltonian cycle (path) problem has various applications.

- Circuit designing (power gating design)
- Software testing (data flow graph)
- Genetic engineering (mapping genome)

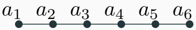
Distance

The distance $d(u, v)$ between u and v is defined as follows:
$d(u, v) =$ length of a shortest path between (u, v), if u and v are connected.

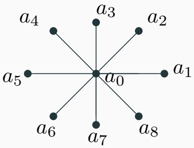
Diameter

The diameter of G, $diam(G)$, is defined by
$diam(G) = \max\{d(u, v) : u, v \in V(G)\}$, if G is connected.
I.e., The diameter of graph is the maximum distance between the pair of vertices.

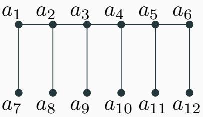
Examples



G_1



G_2



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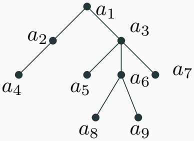
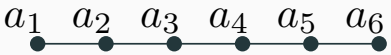
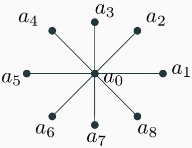


G_3

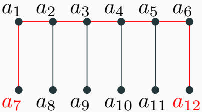
Examples



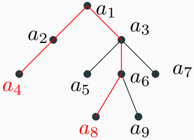
$diam(G_1) = 5$



$diam(G_2) = 2$



$diam(G_3) = 7$



$diam(G_5) = 5$

Special graph classes

A bipartite graph $G(X, Y)$ is called a tree convex bipartite graph with convexity on X if there is an associated tree T on X such that for each vertex u in Y, its neighborhood $N_G(u)$ induces a subtree in T

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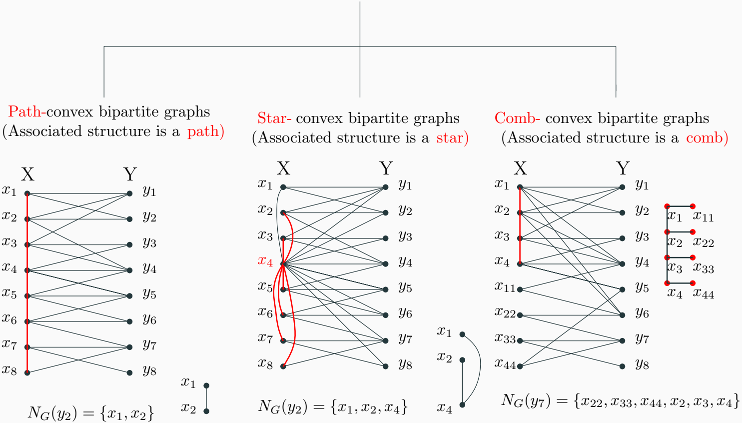
Tree-convex bipartite graphs
(Associated structure is a tree)



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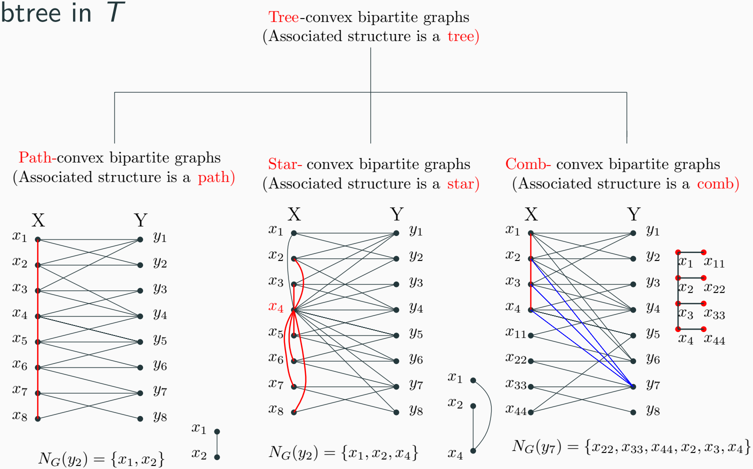

Special graph classes

– It is natural to explore this line of study on graphs having two partitions. A natural choice after bipartite graphs is the class of split graphs.

– G is split $\iff$ $V(G)$ can be partitioned into a clique (K) and an independent set (I)

– A split graph $G(K, I)$ is called a star (comb) convex split graph with convexity on K if there is an associated star (comb) T on K such that for each vertex u in I, its neighborhood $N_G(u)$ induces a subtree in T.

– A split graph $G(K, I)$ is called a star (comb) convex split graph with convexity on I if there is an associated star (comb) T on I such that for each vertex u in K, its neighborhood $N_G^I(u)$ induces a subtree in T.

Literature Survey

An overview

Complexity of Hamiltoian cycle (HC) and Hamiltonian path(HP)

Graph classes	Hamiltonian cycle	Hamiltonian path
Bipartite	NPC [a]	NPC [b]
Split	NPC [b]	NPC [b]

[a] M.S. Krisnamoorthy, 1975 [b] H. Müller et al., 1996

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Star (comb) convex split	?	?

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Our results

Our results

– It is known that HC (HP) is NPC on star convex bipartite.

– We wish to perform a fine-grained analysis of the complexity of HC (HP) with respect to a graph parameter.

– Diameter is one of the popular parameters

– We study the structural and algorithmic aspects of star convex bipartite graphs with respect to the diameter

Results on star convex bipartite graphs

Structural results:

- For a connected star convex bipartite graphs with convexity on X (Y), $diam(G)$ is at most 6.
- A graph G is star convex bipartite with diameter at most 2 if and only if G is complete bipartite.
- Let $G(X, Y)$ be a star convex bipartite graph. If $diam(G)$ is 5, then there exists at least one pendant vertex in Y.
- Let $G(X, Y)$ be a star convex bipartite graph. If $diam(G)$ is 6, then there exists at least two pendant vertices in Y.

Results on star convex bipartite graphs

Complexity of HAMILTOIAN CYCLE (HC) AND HAMILTONIAN PATH(HP)

Problems	Star convex bipartite graphs with diameter k				
	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
Hamiltonian cycle					
Hamiltonian path					

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Results on star convex bipartite graphs

Theorem 1

For star convex bipartite graphs with diameter 3, the Hamiltonian cycle problem is NP-complete.

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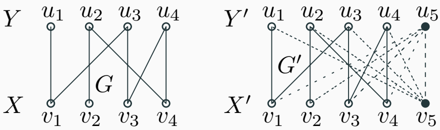
Construction



Mapping: HP in bipartite graphs $\Rightarrow$ HC in star convex bipartite graph with diameter 3

Results on star convex bipartite graphs

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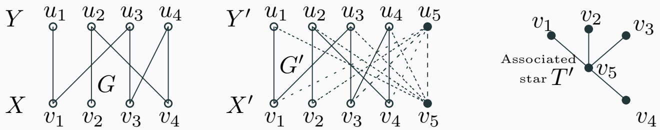

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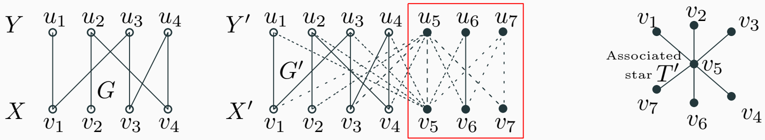

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Results on star convex bipartite graphs

Theorem 2

For star convex bipartite graphs with diameter 3, the Hamiltonian path problem is NP-complete.

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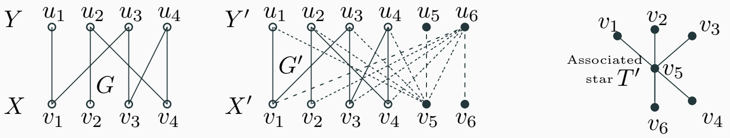

Mapping: HP in bipartite graphs ⇒ HP in star convex bipartite graph with diameter 3

Results on star convex bipartite graphs

Theorem 3
For star convex bipartite graphs with diameter 6, the Hamiltonian path problem is NP-complete.

Results on star convex bipartite graphs

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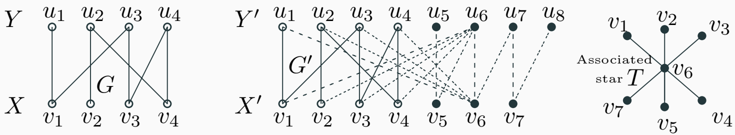
For star convex bipartite graphs with diameter 6, the Hamiltonian path problem is NP-complete.

Construction



Mapping: HP in bipartite graphs ⇒ HP in star convex bipartite graph with diameter 6

Results on star convex bipartite graphs

Complexity of HAMILTOIAN CYCLE (HC) AND HAMILTONIAN PATH(HP)

Problems	Star convex bipartite graphs with diameter k				
	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
Hamiltonian cycle	P	NPC	NPC [a]	P	P
Hamiltonian path	P	NPC	NPC [a]	NPC	NPC

[a]-Chen H et al., 2016

Results on star (comb) convex split graphs

Results on star (comb) convex split graphs

– We study HC and HP in star convex split graph with convexity on $I(K)$

– We need a specific instance of split graphs with $|K| = |I|$ to establish a dichotomy for HC in star convex split graphs with convexity on I.

– Also we look at other cases - For HC we consider the case $|K| > |I|$ and for HP we consider $|K| = |I|$ and $|K| > |I|$.

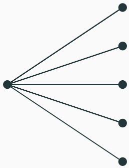
Results on star (comb) convex split graphs

Theorem 4

For star convex split graphs $G(K, I)$ with convexity on I, the Hamiltonian cycle problem is NP-complete.

What next?

$$\underbrace{\underbrace{Split + Star\ convexity}_{NPC} + forbidden\ structure\ (K_{1,5})}_{?}$$



$K_{1,5}$

Results on star (comb) convex split graphs

Lemma 5

Let G be a $K_{1,3}$-free star convex split graph with convexity on I. If $\Delta_G^I = 2$ then $|I| \leq 3$.

Lemma 6

Let G be a split graph. G is $K_{1,3}$-free star convex split graph with convexity on I if and only if one of the following conditions holds.

1. $\Delta_G^I \leq 1$
2. *If there exists a vertex $u \in K$ such that $d_G^I(u) = 2$, then for all $v \in K$, $N_G^I(u) \cap N_G^I(v) \neq \emptyset$*

Results on star (comb) convex split graphs

Observation 1

Let G be a $K_{1,3}$-free star convex split graph with convexity on I. Then for all $u, v \in K$, $N_G^I(u) \cap N_G^I(v) \leq 2$.

Lemma 7

Let G be a $K_{1,4}$-free star convex split graph with convexity on I. For any $u \in K$, the graph H induced on the vertex set $V(G) \setminus N_G^I(u)$ is a l-split graph for some $0 \leq l \leq 2$.

Lemma 8

Let G be a $K_{1,5}$-free star convex split graph with convexity on I. For any $u \in K$, the graph H induced on the vertex set $V(G) \setminus N_G^I(u)$ is a l-split graph for some $0 \leq l \leq 3$.

Results on star (comb) convex split graphs

Theorem 9

For $K_{1,5}$-free star convex split graphs with convexity on I, the Hamiltonian cycle problem is polynomial-time solvable.

Sketch: Since G is $K_{1,5}$-free star convex split, we have the following five cases (1) $\Delta_G^I = 0$, (2) $\Delta_G^I = 1$ (3) $\Delta_G^I = 2$, (4) $\Delta_G^I = 3$, and (5) $\Delta_G^I = 4$.

Results on star (comb) convex split graphs

Theorem 9

For $K_{1,5}$-free star convex split graphs with convexity on I, the Hamiltonian cycle problem is polynomial-time solvable.

Sketch: Since G is $K_{1,5}$-free star convex split, we have the following five cases (1) $\Delta_G^I = 0$, (2) $\Delta_G^I = 1$ (3) $\Delta_G^I = 2$, (4) $\Delta_G^I = 3$, and (5) $\Delta_G^I = 4$.

Case 4: $\Delta_G^I = 3$

- For a star convex split graph G with $\Delta^I = 3$, let $v \in K$, $d_I(v) = 3$, and $U = N^I(v)$. If G is $K_{1,5}$-free, then $|N(U)| \geq |K|/2$.
- For a $K_{1,5}$-free split graph G with $\Delta^I = 3$, let $v \in K$ such that $d^I(v) = 3$, and the split graph $H = G - N^I(v)$. Then, $\Delta_H^I \leq 2$

Results on star (comb) convex split graphs

Short cycle in a $K_{1,5}$-free star convex split graph G

Consider the subgraph H of G where

$X = \{u \in I \mid d(u) = 2\}$, $Y = N(X)$, $V(H) = X \cup Y$ and $E(H) = \{\{u, v\} \mid u \in X; v \in Y\}$.

Clearly, H is a bipartite subgraph of G.

Let C be an induced cycle in H such that $V(K) \setminus V(C) \neq \emptyset$. We refer to C in H as a short cycle in G.

- Let G be a $K_{1,5}$-free star convex split graph with $\Delta^I = 3$. Then G has a Hamiltonian cycle if and only if there are no short cycles in G

Results on star (comb) convex split graphs

Theorem 10

For comb convex split graphs $G(K, I)$ with convexity on K, the Hamiltonian path problem is NP-complete.

Results on star (comb) convex split graphs

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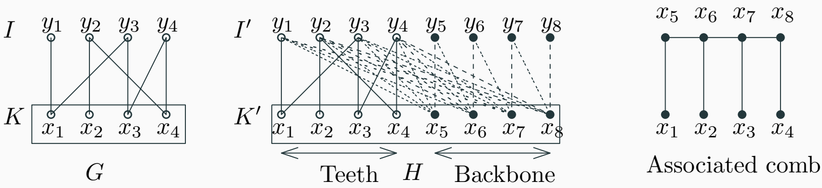

Mapping: HP in split graph $\Rightarrow$ HP in star convex split graph with convexity on K

Summary of our results

Graph classes	Problems	Ordering on K	Ordering on I
Star convex split graphs	H.Cycle	NPC	NPC
	H.Path	NPC	NPC
Comb convex split graphs	H.Cycle	NPC	NPC
	H.Path	NPC	NPC

Summary of our results

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$$\underbrace{\underbrace{Split + Star\ convexity}_{NPC} + forbidden\ structure\ (K_{1,5})}_{HC-is\ polynomial-time\ solvable}$$

Domination and its variants

- A set D is a dominating set in G if every vertex of G is either in D or has a neighbor in D.
- A dominating set is connected (CDS), paired (PDS), and total (TDS), respectively, if the set induces a connected subgraph, induces a perfect matching, and induces a subgraph with no isolated vertex, respectively.

Cont'd.

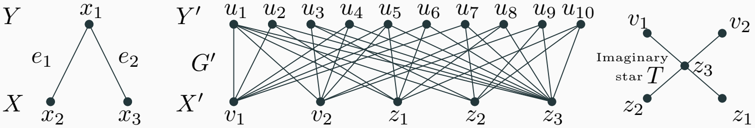
Classical complexity

- Chen et al.,(2016) showed that the domination and its variants are NP-complete on star convex bipartite graphs with diameter at most 4.
- Having shown that the diameter of a star convex bipartite graph is at most six, it is natural to study the complexity of domination on star convex bipartite graphs with diameter k, $2 \leq k \leq 6$.
- For star convex bipartite graphs with diameter 2, the domination and its variants are polynomial-time solvable.

Cont'd.

- We show that the domination is NP-complete on diameter 3 star convex bipartite graphs, which strengthens the result of Chen et al.,

Construction



An example reduction instance G' - star convex bipartite graph with diameter 3 and convexity on X'.

Cont'd.

– We use a construction similar to the construction presented for star convex bipartite graphs with diameter three to prove the following results.

1. Domination is NP-complete on star convex bipartite graphs with diameter $k = 5$, and 6.
2. Domination variants (CDS/PDS/TDS) are also NP-complete on star convex bipartite graphs with diameter k, $3 \leq k \leq 6$.

Cont'd.

Parameterized complexity

- The parameterized version of the domination problem with solution size k as the parameter is defined below:
- **Instance:** A graph G.
  Parameter: A positive integer k.
  Question: Does G have a dominating set $D \subseteq (G)$ such that $|D| \leq k$.

Cont'd.

- We prove that the parameterized version of the domination problem is W[2]-hard (not fixed-parameter tractable - FPT) with parameter being solution size on star convex bipartite graphs.
- If the degree of the star is bounded then we obtain a FPT algorithm.

Conclusions & Open Problems

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We made an attempt to reduce the gap between P vs. NPC for the problems Hamiltonian cycle (path) and Domination problem for the following graphs
(i) Star convex bipartite graphs with diameter as a parameter
(ii) Star (comb) convex split graphs

One can look at the other convex ordering on these graph classes

- Path convex
- Triad convex
- Circular convex

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References

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Thank you

Questions?