Semi-total Domination in Unit Disk Graphs

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Outline

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1. Introduction
   - Preliminaries
   - Problem
   - Related Works
2. Our Problem
3. Our Result
   - NP-complete
   - Approximation Algorithm
4. Conclusion

Dominating Set

Definition 1

A dominating set (DS) of a graph $G = (V, E)$ *is a set* $D \subseteq V$ *such that every vertex* $u \in V$ *is either in* D *or is adjacent to a vertex in* D.

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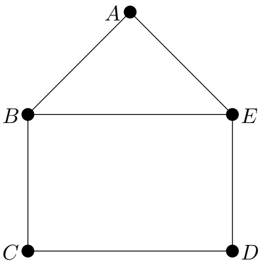

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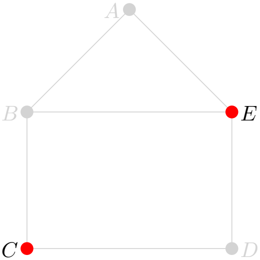

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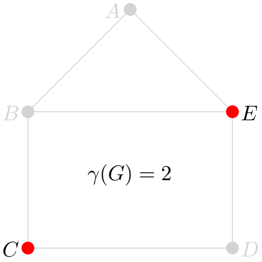

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Total Dominating Set

Definition 2

A total dominating set (TDS) of a graph $G = (V, E)$ *is a set* $D_t \subseteq V$ *such that* (i) D_t *is a dominating set of* G *(domination property) and* (ii) $G[D_t]$ *does not have any isolated vertex (total property).*

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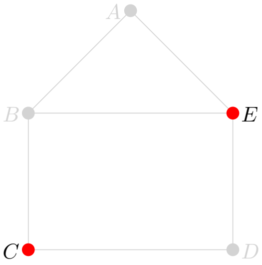

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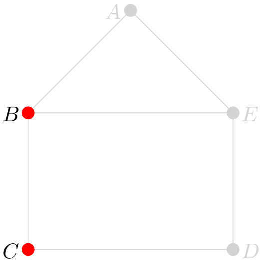

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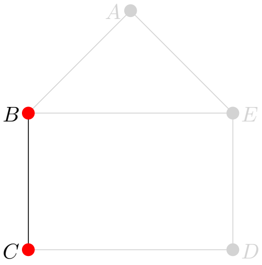

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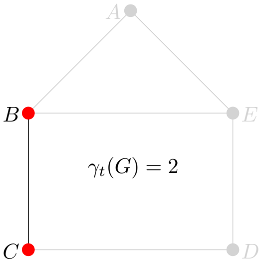

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Semi-total Dominating Set

Definition 3

A Semi-total dominating set (T2DS) of a graph $G = (V, E)$ is a set $D_{t2} \subseteq V$ such that (i) D_{t2} is a dominating set (domination property), and (ii) for each $u \in D_{t2}$, there exists a vertex $v \in D_{t2}$ such that $d(u, v) \leq 2$ (semi-total property).

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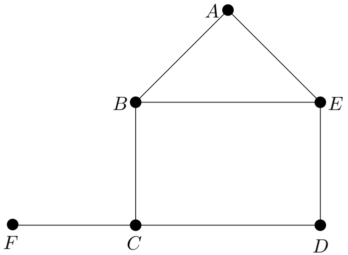

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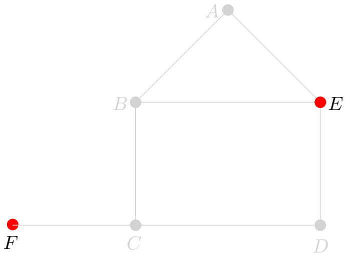

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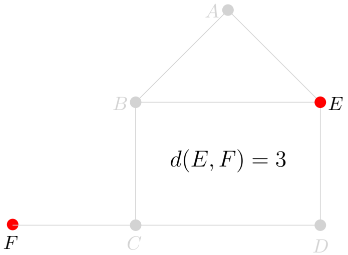

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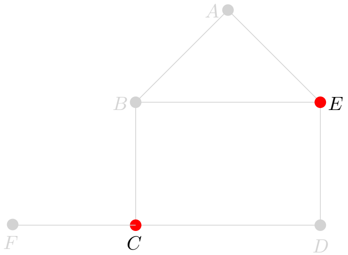

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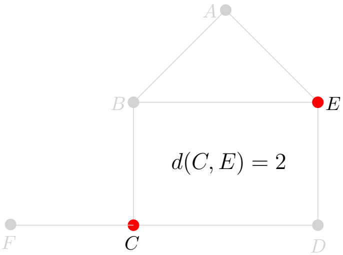

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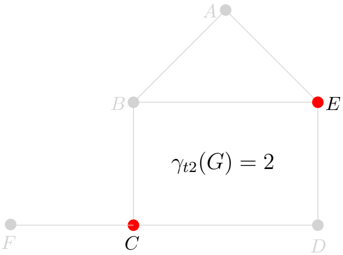


Figure 3

Unit Disk Graph (UDG)

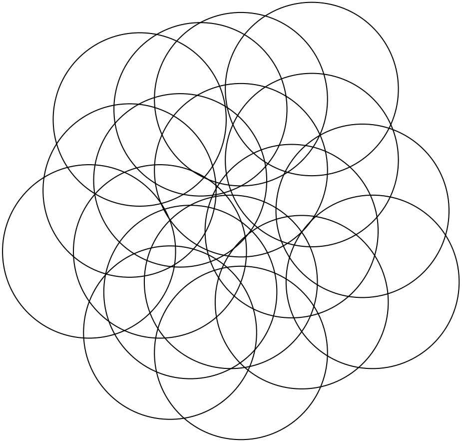
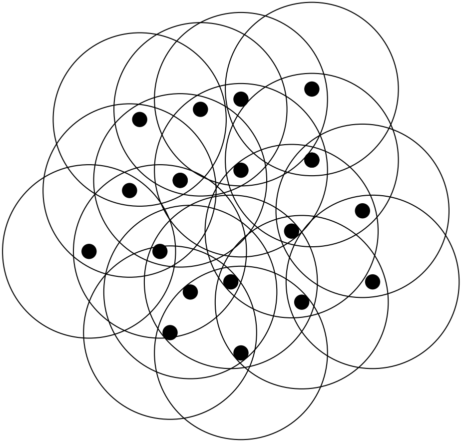


Figure 4

Unit Disk Graph (UDG)



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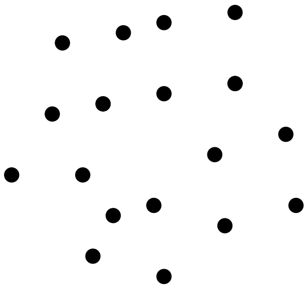

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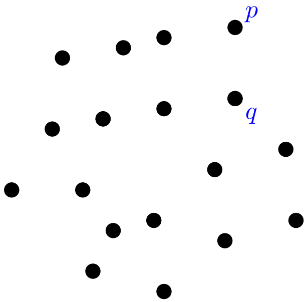

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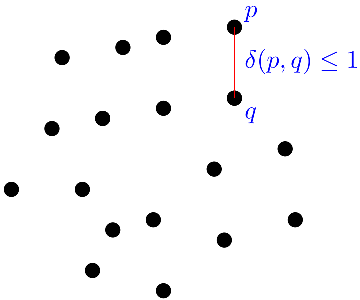

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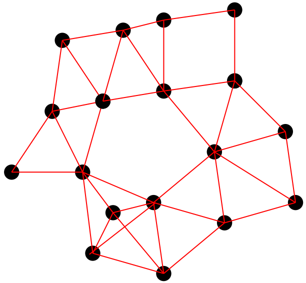

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Semi-total Dominating Set Problem

T2DS Problem: Given a graph $G = (V, E)$, the problem finds a semi-total dominating set (T2DS) $D_{t2} \subseteq V$ of minimum size.

Outline

Related Works

1. First introduced by Goddard et al., $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$ [Goddard et al., 2014].
2. Connected graph with at least 2 vertices: $\gamma_{t2}(G) \leq \alpha'(G)$ [Henning and Marcon, 2014].
3. Connected graph with $n(\geq 4)$ vertices: $\gamma_{t2} \leq \frac{n}{2}$ for trees [Marcon, 2015].
4. Connected claw-free cubic graph: $\gamma_{t2} \leq \frac{n}{2}$ [Henning and Marcon, 2016].
5. NP-complete: planar graphs, chordal bipartite graphs and split graphs, $2 + 3\ln(\Delta + 1)$ - factor approximation algorithm for the semi-total DS problem in general graphs [Henning and Pandey, 2019].
6. NP-complete to recognize the graphs that satisfy $\gamma(G) = \gamma_{t2}(G)$ and $\gamma_t(G) = \gamma_{t2}(G)$ [Galby et al., 2020].

The T2DS Problem in UDGs

T2DS-UDGs: Given a geometric unit disk graph $G = (V, E)$, the objective of the minimum semi-total dominating set problem is to find a semi-total dominatiing set of minimum size.

Our Result

1. The T2DS-UDGs problem is NP-complete.
2. 6-factor approximation algorithm (UDGs)
3. $2 + \ln(\mathbb{D} + 1)$-factor approximation algorithm (General graphs)

Outline

NP-complete

1. Decision version of semi-total dominating set problem in UDGs (D-T2DS-UDGs problem)

   Instance: A UDG $G = (V, E)$ defined on a point set P and a positive integer $k \leq |V|$.
   Question: Does there exist a semi-total dominating set (of G) of size at most k?

2. Decision version of vertex cover problem in planar graphs of degree at most 3 (D-VC-PGD3 problem) [Lichtenstein, 1982]

   Instance: A planar graph of degree at most 3 and a positive integer $k \leq |V|$.
   Question: Does there exist a vertex cover (of G) of size at most k?

Reduction

1. $D\text{-}VC\text{-}PGD3 \rightarrow D\text{-}T2DS\text{-}UDGs$
2. Graph Construction: $G(V, E) \rightarrow G'(V', E')$
3. Claim: G has a vertex cover of size at most k if and only if G' has a semi-total dominating set of size at most $k + 2n + 2\ell$.

Graph Construction

1. $G(V, E)$: Planar graph with degree at most 3
   1. Embedding
   2. Inclusion of auxiliary points
   3. Inclusion of gadgets
2. $G'(V', E')$: UDG

Embedding

Lemma 1

[Valiant, 1981] Let $G = (V, E)$ *be a planar graph of degree at most* 3*. The graph* G *can be embedded in a grid of area* $O(|V|^2)$ *such that each* $v \in V$ *lies in a grid point with co-ordinate* $(5i, 5j)$*, where* i *and* j *are integers and each edge* $e \in E$ *is a finite sequence of consecutive segments of length* 5 *units along the grid lines.*

Embedding

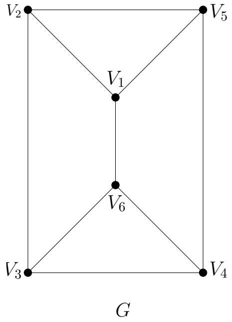


Figure 5

Embedding

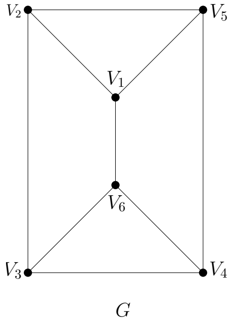
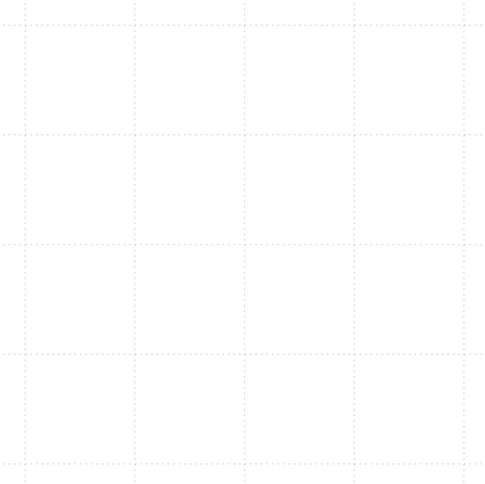
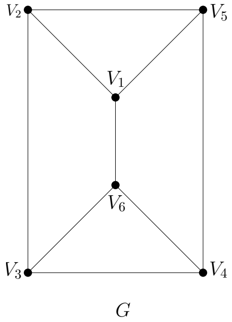


Figure 6



5×5 Grid

Embedding



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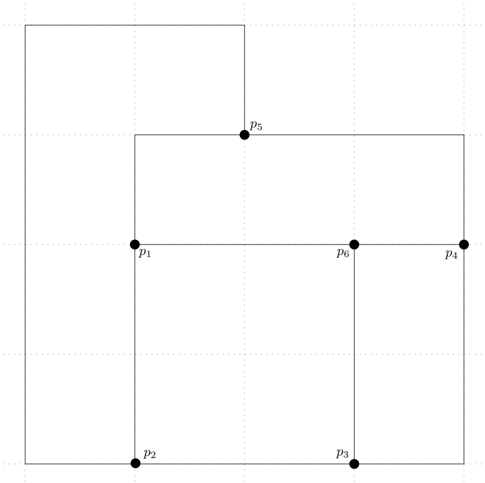

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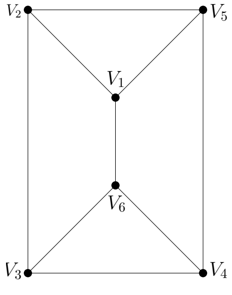

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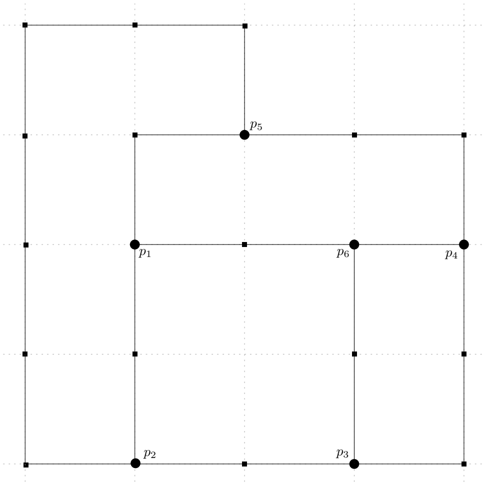

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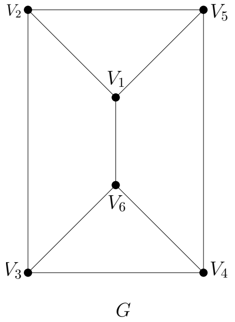

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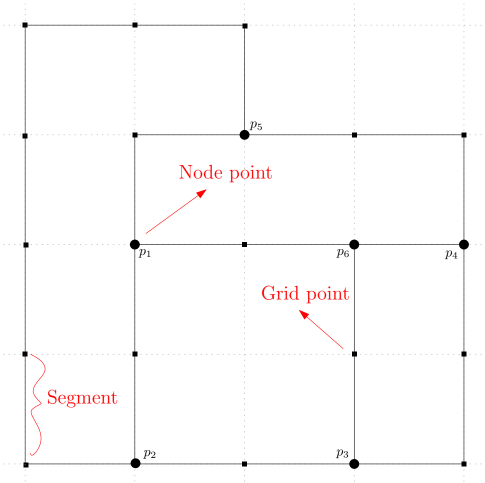
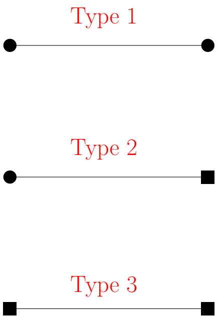


5×5 Grid

Inclusion of Auxiliary Points



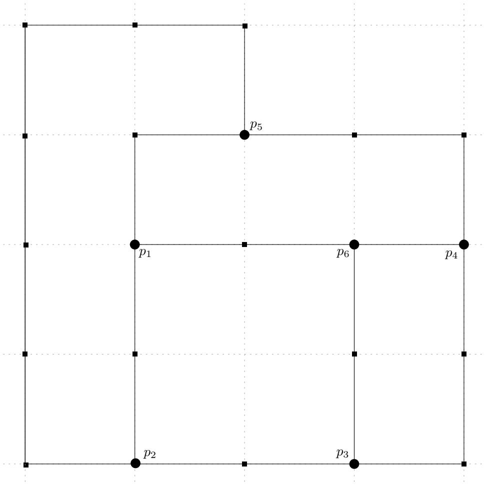
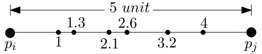
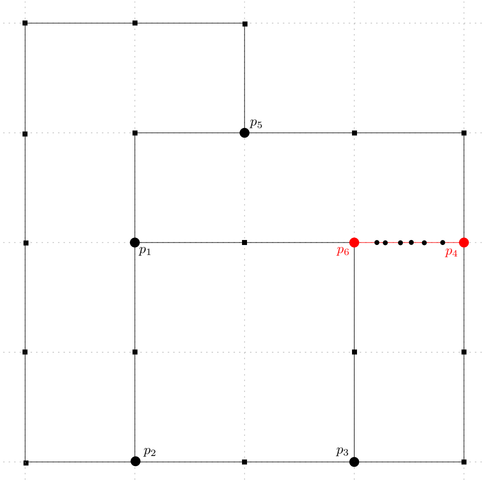


Figure 7

Inclusion of Auxiliary Points





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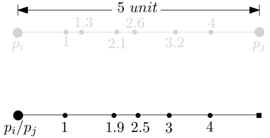

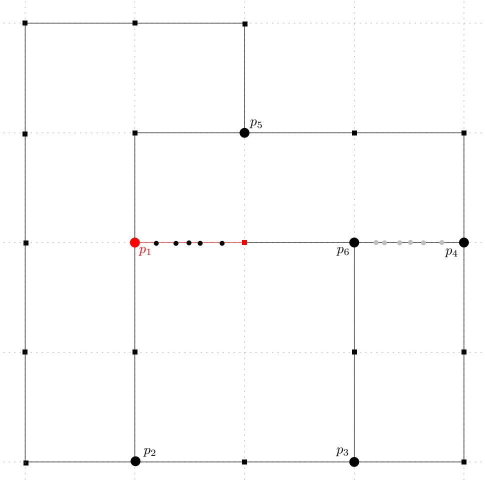

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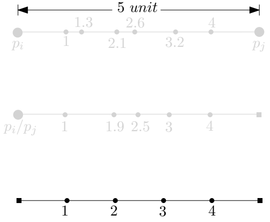

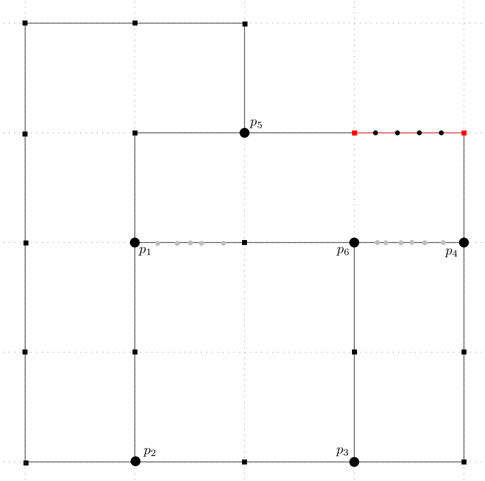

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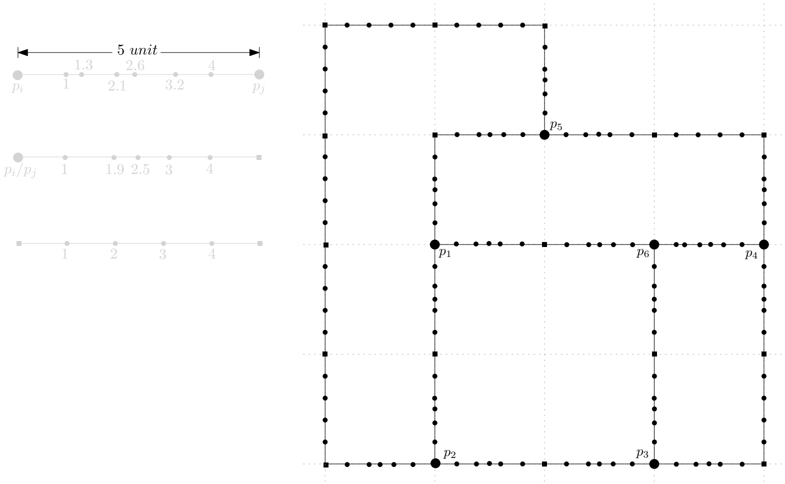


Figure 7

Inclusion of Gadgets

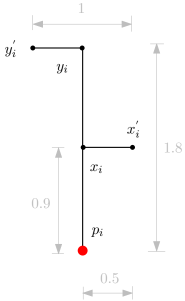


Figure 8: Gadget

Inclusion of Gadgets

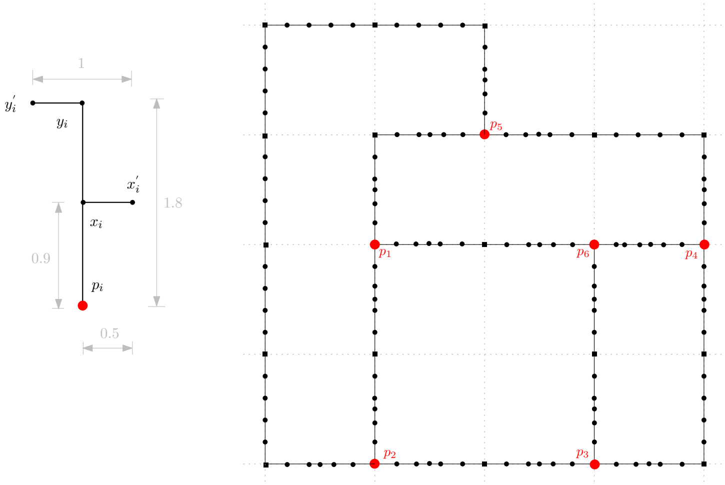


Figure 9

Inclusion of Gadgets

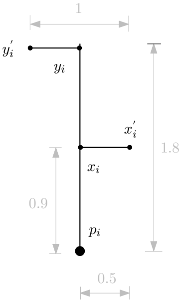
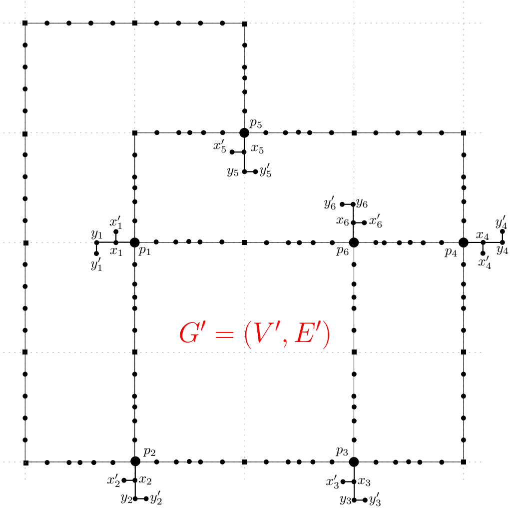


Figure 9



Claim: Graph $G' = (V', E')$ is a UDG.

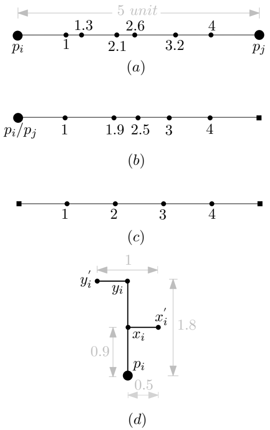
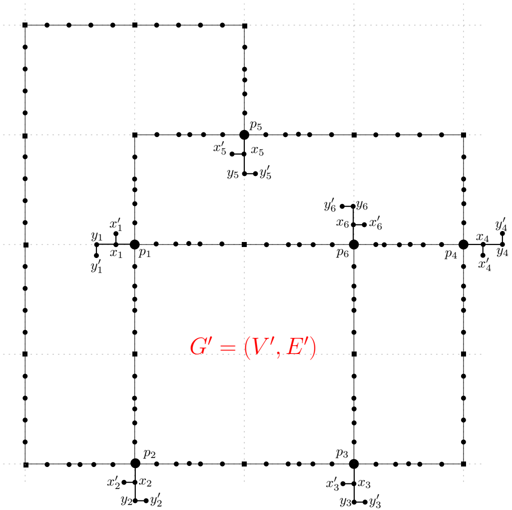


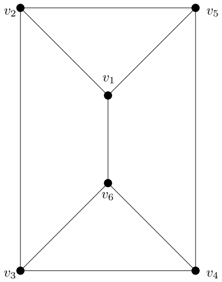
Figure 10



Reduction cont.

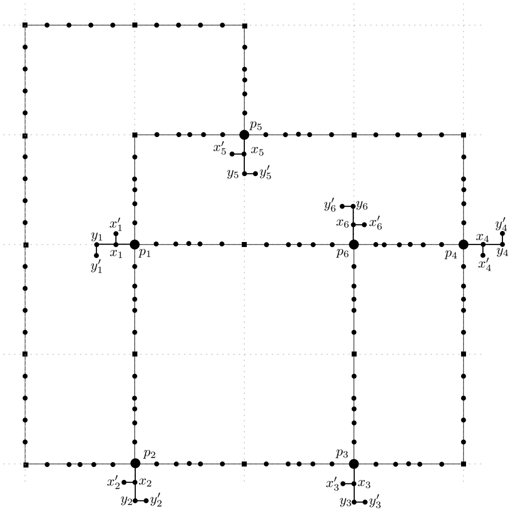
Claim: G has a vertex cover of size at most k if and only if G' has a semi-total dominating set of size at most $k + 2n + 2\ell$.

Necessity:



$G = (V, E)$

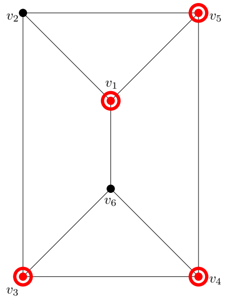
Figure 11



$G' = (V', E')$

Necessity:

$VC(G) \leq k$



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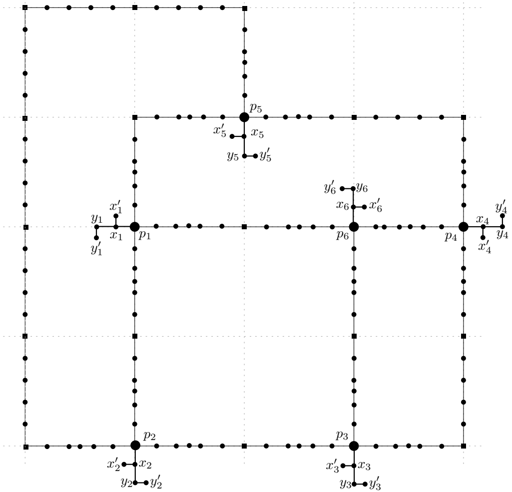

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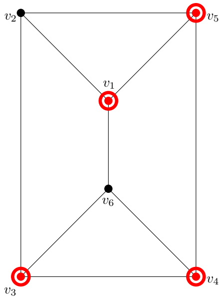


$G = (V, E)$

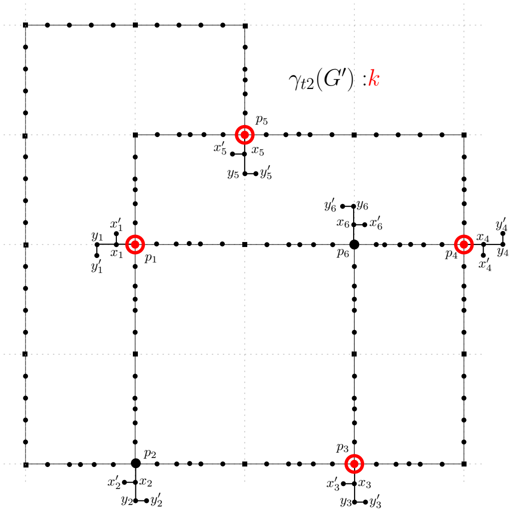
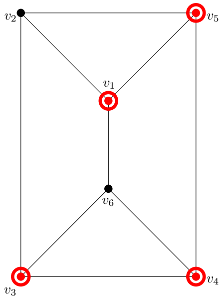


Figure 11

$G' = (V', E')$

Necessity:

$VC(G) \leq k$



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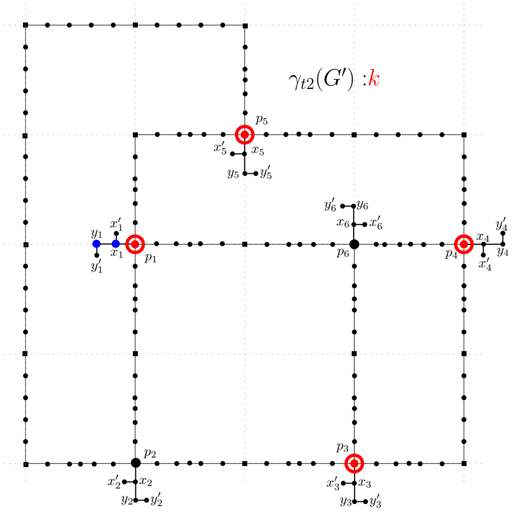

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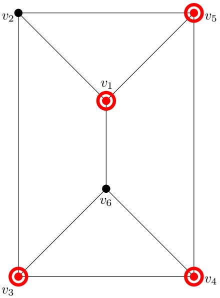

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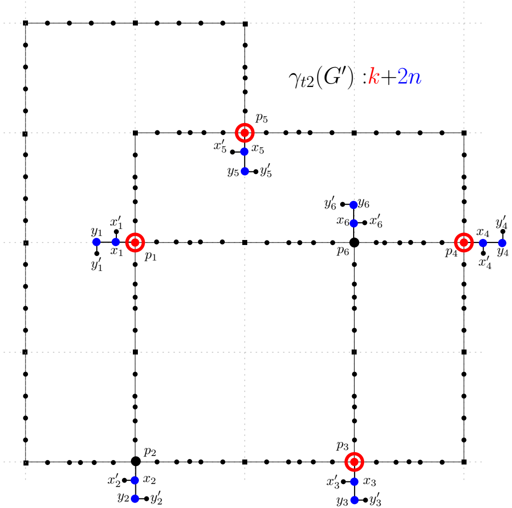

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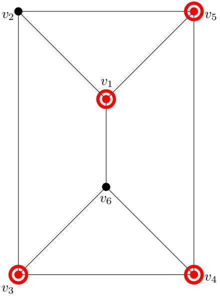

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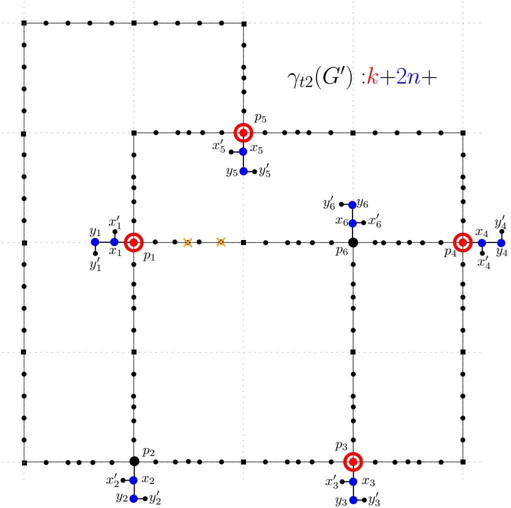

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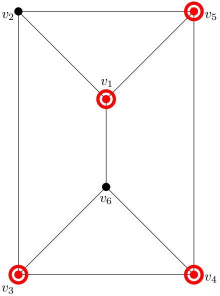

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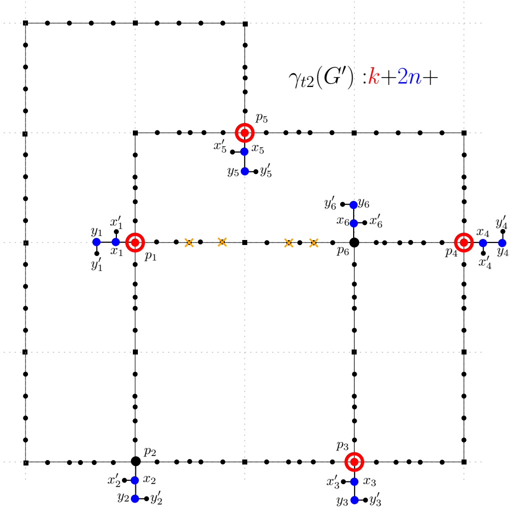

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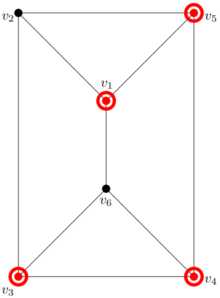
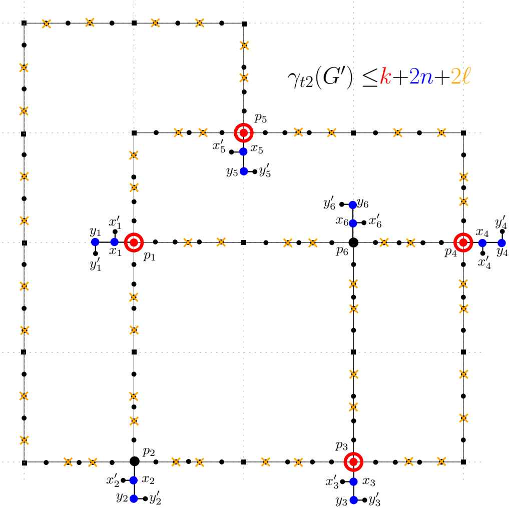


$G = (V, E)$

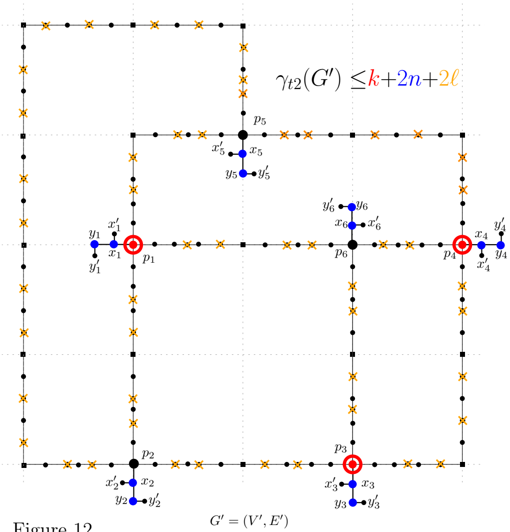
Figure 11

$\gamma_{t2}(G') \leq k + 2n + 2\ell$

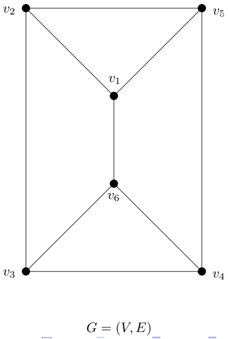


$G' = (V', E')$

Sufficiency:



$VC(G) \leq k$



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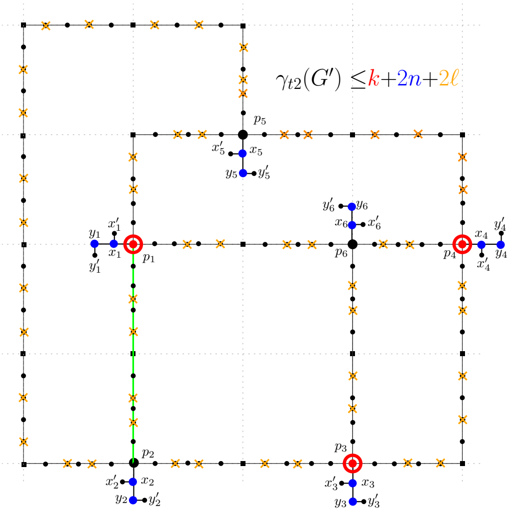

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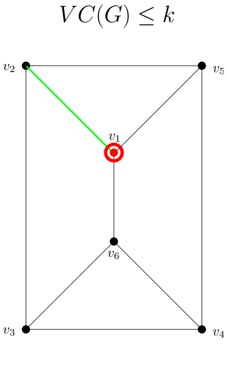

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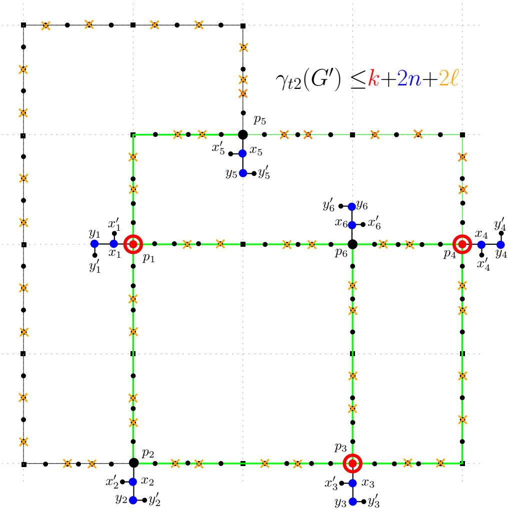

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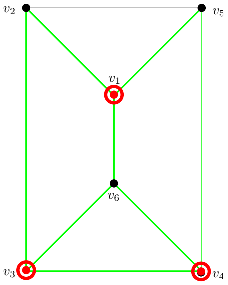

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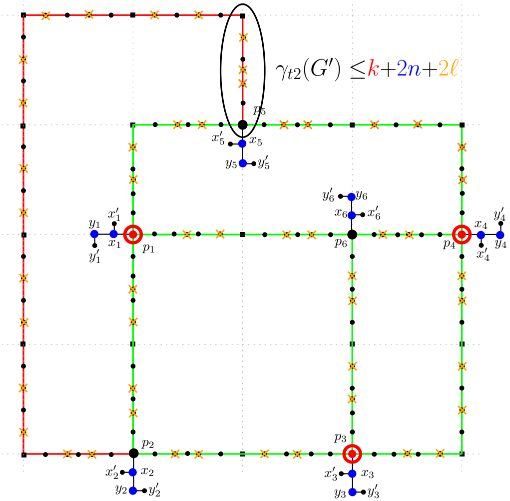

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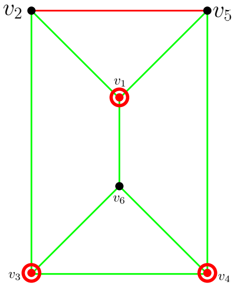

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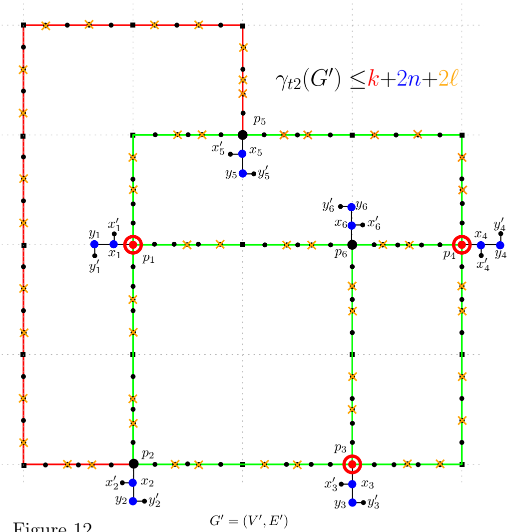

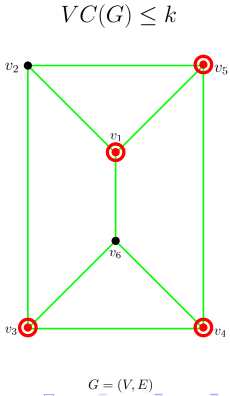

Figure 12

Outline

6 - *factor* Approximation Algorithm (UDGs)

Algorithm 1

Input: a UDG $G = (V, E)$

Output: a semi-total dominating set D_{t2}

- Finds a Maximal Independent Set D (domination Property)
- Finds a set T (semi-total Property)
- Report $D_{t2} = D \cup T$

Algorithm 1

- Finds a Maximal Independent Set D
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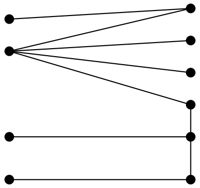

Figure 12: $G(V, E)$

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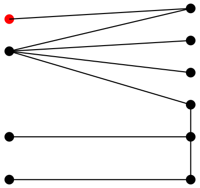

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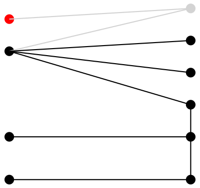

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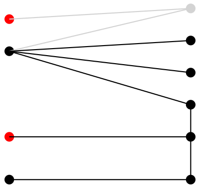

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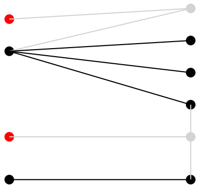

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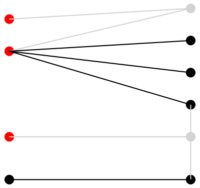

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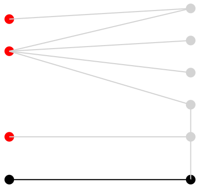

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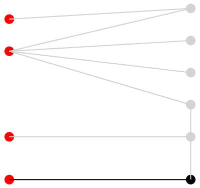

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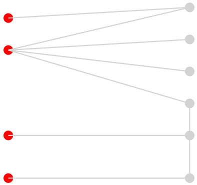

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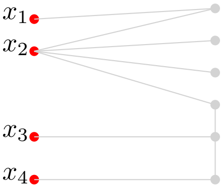

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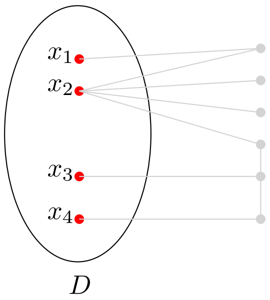

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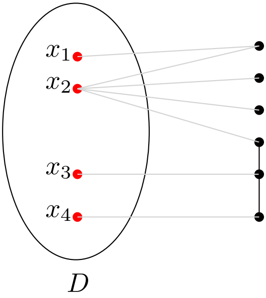

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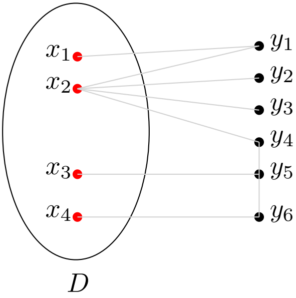

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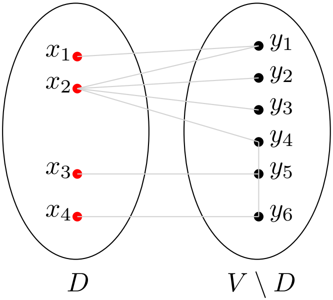

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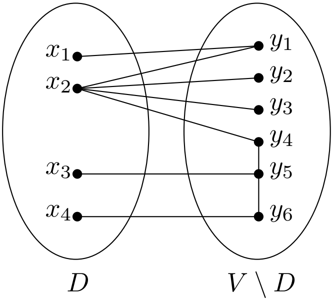

Figure 13: $G(V, E)$

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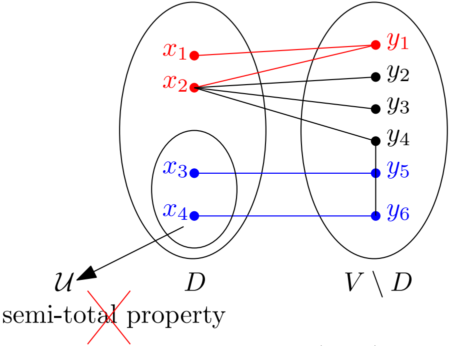

Figure 13: $G(V, E)$

$$
\begin{array}{l}
\textit{for each } u \in V \quad \textit{do} \\
\qquad S_u = N_G(u) \cap D \\
\qquad \textit{if} \quad |S_u| > 1 \quad \textit{then} \\
\qquad\qquad X = X \cup S_u \\
\mathcal{U} = D \setminus X
\end{array}
$$

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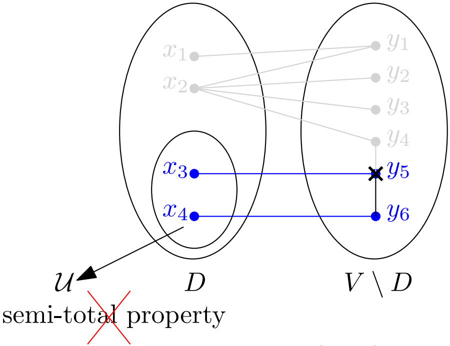

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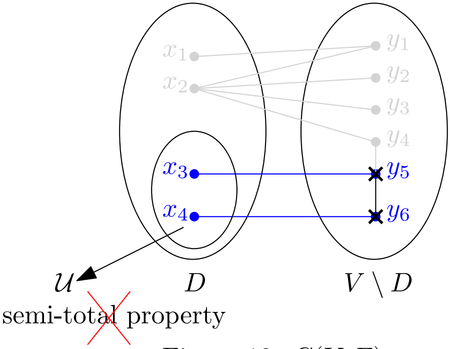

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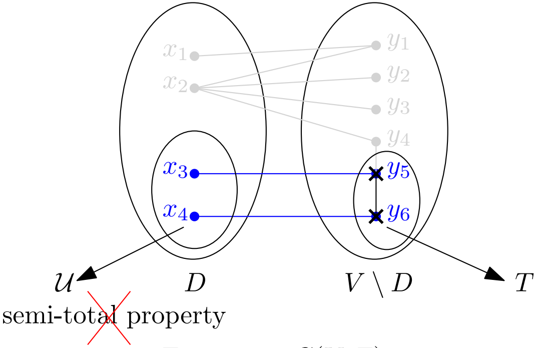

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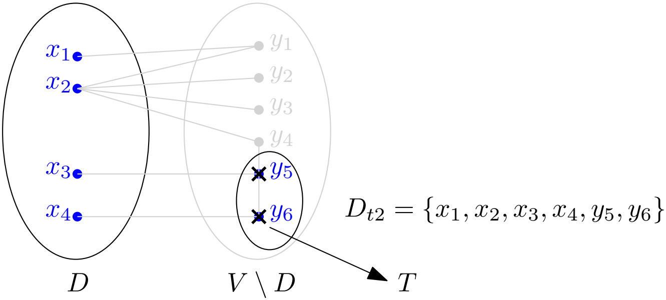
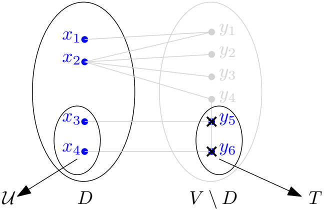


Figure 13: $G(V, E)$

Analysis

Lemma 2
If D^* *is an optimal dominating set of* G*, then* $|T| \leq |D^*|$.



Proof. Assume $|T| > |D^*| \implies |\mathcal{U}| > |D^*|$
$\exists u \in D^*$ which dominates 2 or more vertices in $\mathcal{U}$.
This leads to a contradiction that there is no vertex $v \in V$ that has more than one neighbor in $\mathcal{U}$.

Analysis

$$
\begin{aligned}
|D_{t2}| &= |D \cup T| \\
&\leq |D| + |T| \\
&\leq 5|D^*| + |D^*| \text{ (Lemma 3 and Lemma 2)} \\
&= 6|D^*| \\
&\leq 6|D^*_{t2}| \text{ (Observation 1)}
\end{aligned}
$$

Lemma 3 [Marathe et al., 1995]
Let $\mathcal{P}$ be a unit disk centered at point p and let S be a set of independent unit disks such that each disk in S contains the point p, then $|S| \leq 5$.

Observation 1 [Goddard et al., 2014]
For a given graph G, $\gamma(G) \leq \gamma_{t2}(G)$.

Approximation Algorithm (General Graphs)

Algorithm 2

- Phase 1: Use Lemma 4 to find a DS D of G
- Phase 2: Find a set T as in *Algorithm* 1

Analysis:
$$\begin{aligned} |D_{t2}| &= |D \cup T| \\ &\leq |D| + |T| \\ &\leq (1 + \ln(\mathbb{D}+1))|D^*| + |D^*| \text{ (Lemma 4 and Lemma 2)} \\ &= (2 + \ln(\mathbb{D}+1))|D^*| \\ &\leq (2 + \ln(\mathbb{D}+1))|D^*_{t2}| \text{ (Observation 1)} \end{aligned}$$

Lemma 4 [Klasing and Laforest, 2004]
The minimum domination problem in a graph with maximum degree $\mathbb{D}$ *can be approximated with an approximation ratio* $1 + \ln(\mathbb{D}+1)$.

Conclusion

- Semi-total DS problem is NP-complete in UDGs.
- 6-factor approximation algorithm for UDGs.
- $(2 + \ln(\mathbb{D} + 1))$-factor approximation algorithm for General graphs (improvement over $(2 + 3\ln(\mathbb{D} + 1))$).

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Thank You