Total Coloring of Some Graph Operations

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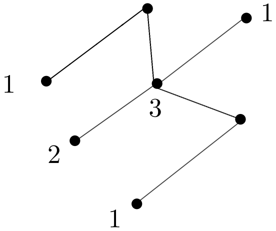
Outline of the Presentation

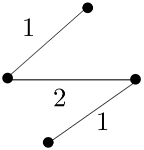
Priliminary Definitions

- **Total coloring** of a graph is an assignment of colors to both the vertices and edges such that no two adjacent or incident elements receive the same color.
- A total coloring using at most k colors is called a k**- total coloring**.
- The least number of colors needed for the proper total coloring of G is the **total chromatic number of G**, denoted as $\chi_T(G)$.

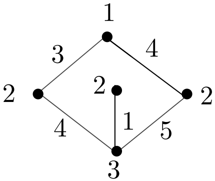
Priliminary Definitions



vertex coloring



edge coloring



total coloring

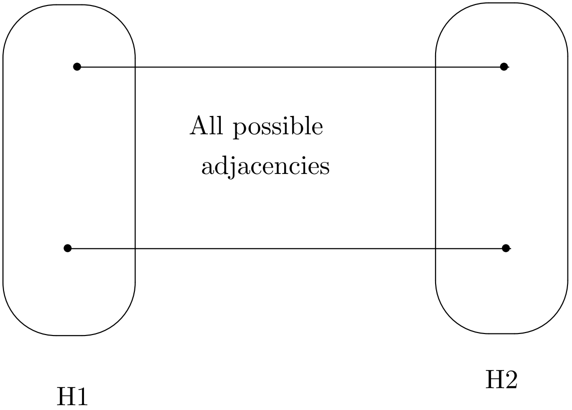
- During 1960's Behzad and Vizing independently raised the following conjecture:

Priliminary Definitions

"*Every graph is total colorable using its maximum degree plus two colors.*" It is known as the **Total Coloring Conjecture (TCC)**, which is one among the classic open problems in graph theory.

- Even though many well-known researchers from different parts of the world have studied TCC for over 60 years, it remains open till now.

- Let G be a graph with vertex set $V(G) = \{1, 2, \ldots, k\}$ and $H_1, H_2, \ldots, H_k$ be the collection of graphs. The **G-generalized join** of $H_1, H_2, \ldots, H_k$, denoted by $G[H_1, H_2, \ldots, H_k]$, is the graph G' with vertex set $V(G') = \bigcup_{i=1}^{k} V(H_i)$ and edge set
$$E(G') = \left(\bigcup_{i=1}^{k} E(H_i)\right) \cup \left(\bigcup_{ij \in E(G)} \{xy | x \in V(H_i), y \in V(H_j)\}\right).$$
- If $H_i \cong H$ for $1 \leq i \leq k$, then $G[H, H, \ldots, H]$ is the standard **lexicographic product of G and H** and it is denoted as $G \circ H$. If $G = K_2$, then $K_2[H_1, H_2]$ is the well known **join** of graphs H_1 and H_2 and it is denoted by $H_1 \vee H_2$.


All possible
adjacencies
H1
H2

TCC : Some known Results

- Total Coloring Conjecture (TCC) : *For any graph G*, $\chi_T(G) \leq \Delta(G) + 2$.
- The graphs having a total coloring using $\Delta(G) + 1$ colors are called **type-1 graphs** and those that having a total coloring using minimum $\Delta(G) + 2$ colors are said to be **type-2 graphs**.
- Complete graphs on n vertices, K_n satisfies TCC and is type-1 for n odd and type-2 for n even (Behzad, Chartrand and Cooper).
- Total coloring of the cartesian product of almost all graphs were discussed (Kemnitz, Marangio, Zmazek, Zerovnik).

TCC : Some known Results

- The current researchers go for some relaxed version of TCC which is known as the Weak TCC.
  k-Total coloring Conjecture (k-TCC):
  For any graph G, $\chi_T(G) \leq \Delta(G) + k$, for some fixed positive integer $k \geq 2$.
  The 2-TCC is nothing but the original TCC and 3-TCC is known as the **Weak TCC**.
- Seoud et al. calculated the total chromatic number of the join of two paths. Guanggrong Li and Limin Zhang proved that the join of a complete in-equipartite graph and a path is type-1.

TCC : Some known Results

- R. Vignesh et al. proved the validity of TCC for the join of a graph satisfying TCC with itself. But we found that the existence of a proper edge coloring that is just mentioned in the proof without any proper explanation is not always mandatory. Hence in order to overcome that here we give a rigorous proof using the coloring technique explained in the Lemma which explained here.

TCC : Some known Results

- Even though we do not have a proof for the existence of TCC, we have seen that it is proved for a vast range of graphs. Here we are going to see the same for some graph operations namely the join of graphs and the lexicographic product of graphs.
- We use the following result in our proofs:

Theorem (Konig)

For any bipartite graph, $\chi'(G) = \Delta(G)$.

Theorem

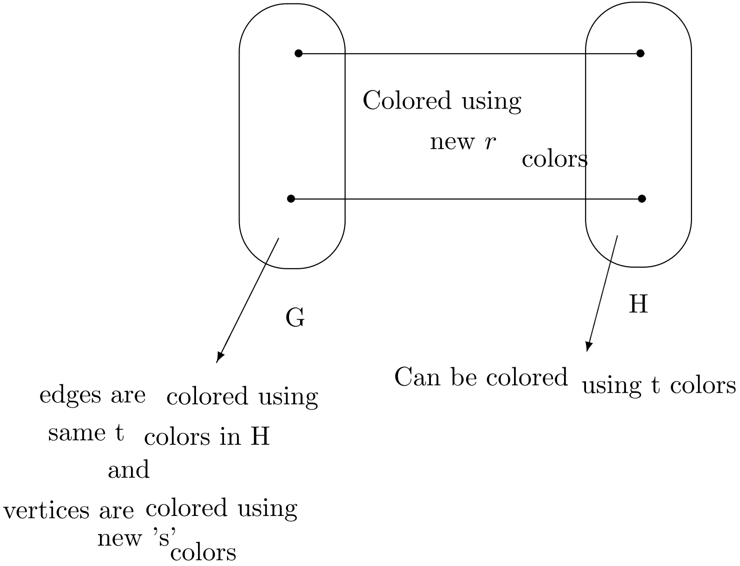
Let G and H be graphs with m and n vertices, respectively. If $\chi'(G) \leq \chi_T(H)$, then

$$max\{\Delta(H)+m, \Delta(G)+n\}+1 \leq \chi_T(G \vee H) \leq max\{m,n\}+\chi_T(H)+\chi(G).$$

In general, $max\{\Delta(H)+m, \Delta(G)+n\}+1 \leq \chi_T(G \vee H) \leq max\{m,n\} + max\{\chi'(G), \chi'(H)\} + \chi(H) + \chi(G).$

Proof

•Let $r = \max\{m, n\}$, $s = \chi(G)$ and $t = \chi_T(H)$.
• Then, $\chi_T(G \vee H) \geq \Delta(G \vee H) + 1 = max\{\Delta(H) + m, \Delta(G) + n\} + 1$. Hence the lower bound.
•For the upper bound, define a coloring of $r + s + t$ colors as follows:
i)Color the edges and vertices of H using t colors.
ii)Then, $t \geq \chi'(G)$. Hence the edges of G can be colored using the same t colors.
iii)Color the vertices of G using new s colors.
iv)the edges between G and H is a bipartite graph of maximum degree r, and hence can be edge colored properly using new r colors.
So $G \vee H$ is $r + s + t$-total colorable. Hence the result.


Colored using
new r colors
G
H
edges are colored using
same t colors in H
and
vertices are colored using
new 's' colors
Can be colored using t colors

Corollary

If G is a bipartite graph and H is a graph satisfying TCC and both having the same maximum degree, then

$$\chi_T(G \vee H) \leq \begin{cases} k+4 & \text{if } H \text{ is type} - 2; \\ k+3 & \text{if } H \text{ is type} - 1, \end{cases} \quad \text{where } k = \Delta(G \vee H).$$

Lemma:

The edge set of $K_{n,n}$ can be partitioned into $n+1$ matchings such that each vertex of $K_{n,n}$ is saturated by n matchings among them.

Proof

• Let the vertex partition be $X = \{u_1, u_2, \ldots, u_n\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$.

• Let $M_0 = \{u_i v_i : 1 \leq i \leq n\}$ and $R_0 = K_{n,n} - M_0$.

• Now, successively define R_j's and M_j's as, for $1 \leq j \leq n-2$,
$R'_j = R_{j-1} - \{u_j, v_j\}, M_j = A_j \cup B_j$, where
$A_j = \{u_i v_{i+j+1(mod\ n)} : 1 \leq i \leq j-1\ or\ i = n\}$,
$B_j = \{u_i v_{i+j(mod\ n)} : j+1 \leq i \leq n-1\}$ and
$R_j = R_{j-1} - M_j$.

• Again, $R'_{n-1} = R_{n-2} - \{u_{n-1}, v_{n-1}\}$,
$M_{n-1} = \{u_i v_{2i+1(mod\ n)} : 1 \leq i \leq n\ and\ i \neq n-1\}$,
$R_{n-1} = R_{n-2} - M_{n-1}$ and
$R'_n = R_{n-1} - \{u_n, v_n\}$,
$M_n = \{u_i v_{2i(mod\ n)} : 1 \leq i \leq n-1\}$.

Proof

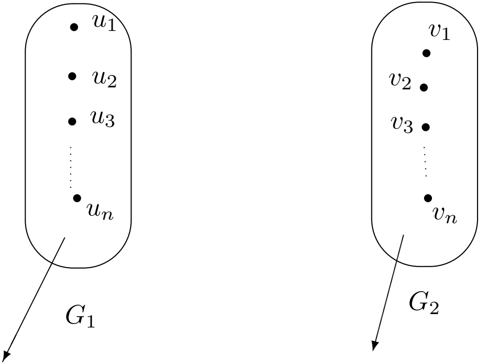
• Then, M_j is a matching in R'_j, for $1 \leq j \leq n$ and each vertex u_j (as well as v_j) in $K_{n,n}$ is M_i-saturated for all $i \in \{1, 2, \ldots, n\} \backslash \{j\}$.
• $|M_0| = n$, $|M_j| = n - 1$ for $1 \leq j \leq n$ and $E(R'_n) \backslash M_n = \emptyset$. Hence $\sum_{j=0}^{n} |M_j| = |E(K_{n,n})|$.
• Finally, we verify that $\{M_j\}_{j=1}^{n}$ are disjoint. Hence the result the follows.

Theorem

If G is a graph satisfying TCC, then $G \vee G$ satisfies TCC.

Proof

•Clearly $G \vee G$ contains two copies of G, say G_1 and G_2. Let $V(G_1) = \{u_1, u_2, \dots, u_n\}$ and $V(G_2) = \{v_1, v_2, \dots, v_n\}$.
•Let $\Delta(G) = k, \Delta(G \vee G) = k + n$ and c be a $k + 2$-total coloring of G using $1, 2, \dots, k + 2$.
•Construct a total coloring of $G \vee G$ using at most $k + n + 2$ colors.
$i)$Color $V(G_1) \cup E(G_1) \cup E(G_2)$ using c.
$ii)$Consider the $n + 1$ partition of the in between edges as explained in the lemma and c_i denotes the missing color in u_i under c.
$iii)$Now, the edges in M_0 is colored using c_i and edges in $M_j \cup \{v_i\}$ is colored using $k + 2 + j$ for $1 \leq j \leq n$. Hence the result.



totally it is colored using
$1, 2, ..k + 2$

vertices are colored using
$k + 3, k + 4, \ldots, k + n + 2$
and
edges are having colors
from $\{1, ..k + 2\}$

Corollary

If a graph G satisfies TCC, then $\bigvee_{i=1}^{m} G_i$ satisfies TCC, where $G_i \cong G$ for $1 \leq i \leq n$ and $m = 2^t$ for any positive integer t.

Theorem

If G and H are two graphs with m and n vertices respectively. Also, $\Delta(G) \geq \Delta(H)$, $m \leq n$ and G satisfies TCC, then $\chi_T(G \vee H) \leq \Delta(G \vee H) + 3$.

Theorem

If G and H are two k-regular graphs with same odd order n, then $G \vee H$ is not type-1.

Corollary

For an odd ordered regular G graph satisfying TCC, the join $G \vee G$ is type-2.

Corollary

For an odd positive integer $m \geq 3$, $C_m \vee C_m$ is a type-2 graph.

Proposition

For $m, n \geq 3$, the join of two cycles $C_m \vee C_n$ satisfies TCC .

Proof

Let $G = C_m \vee C_n$ and also let $m \geq n$. Clearly, $\Delta(G) = m + 2$.
Let $V(C_m) = \{v_1, v_2, v_3, \ldots, v_m\}$ and $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$.
For $m = n = 3$, the result is obvious.
Assume that, $m > 3$ and $n \geq 3$.

Proof

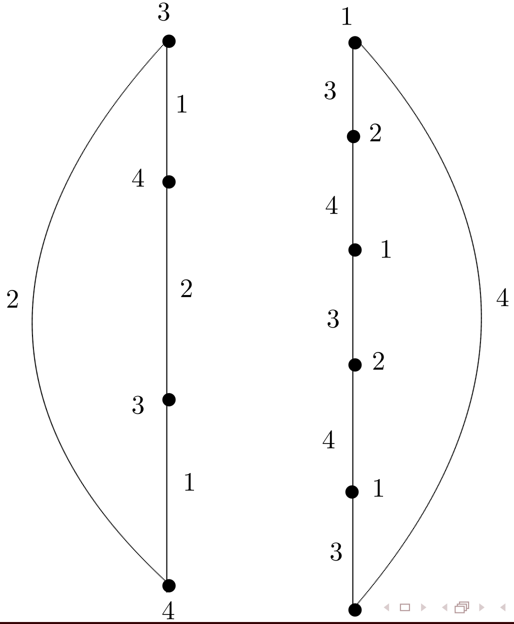
Case 1. m and n are even
The following is a total coloring of G using $m+4$ colors.

$$c(v_i) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \end{cases} \qquad c(u_i) = \begin{cases} 3 & \text{for} \quad i \equiv 1 \mod 2 \\ 4 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

$$c(v_i v_{i+1}) = \begin{cases} 3 & \text{for} \quad i \equiv 1 \mod 2 \\ 4 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

$$c(u_i u_{i+1}) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

Remaining uncolored edges forms a bipartite graph of maximum degree m and it can be properly colored using m new colors.


3
1
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3
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2
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3
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1
1
3
4

Proof continuation

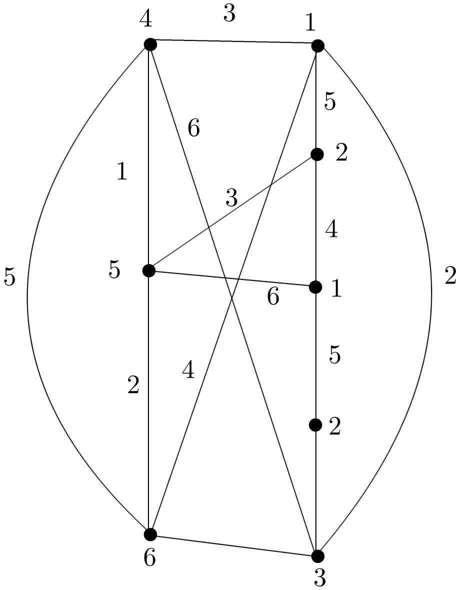
Case 2. m and n are odd

$$c(v_i) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq m \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \\ 3 & \text{for} \quad i = m \end{cases}$$

$$c(u_i) = \begin{cases} 4 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq n \\ 5 & \text{for} \quad i \equiv 0 \mod 2 \\ 6 & \text{for} \quad i = n \end{cases}$$

$$c(v_i v_{i+1}) = \begin{cases} 5 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq m \\ 4 & \text{for} \quad i \equiv 0 \mod 2 \\ 2 & \text{for} \quad i = m, i+1 = 1 \end{cases}$$

$$c(u_i u_{i+1}) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq n \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \\ 5 & \text{for} \quad i = n, i+1 = 1 \end{cases}$$


4
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1
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1
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4
5
2
6
3

Proof continuation

Next we color some of the edges in between C_m and C_n.

$$c(v_iu_j) = \begin{cases} 1 & \text{for} \quad i = m \text{ and } j = n \\ 3 & \text{for} \quad 1 \leq i = j \leq n-1 \\ 4 & \text{for} \quad i = 1 \text{ and } j = n \\ 6 & \text{for} \quad i = m \text{ and } j = 1 \end{cases}$$

Also, for $2 \leq i \leq n-1$, color $c(u_iv_{i+1 \bmod (n-2)}) = 6$.

The remaining uncolored edges forms a bipartite graph, with maximum degree $m-2$ and hence the result follows by Konig's theorem.

Proof continuation

Case 3. m is even and n is odd

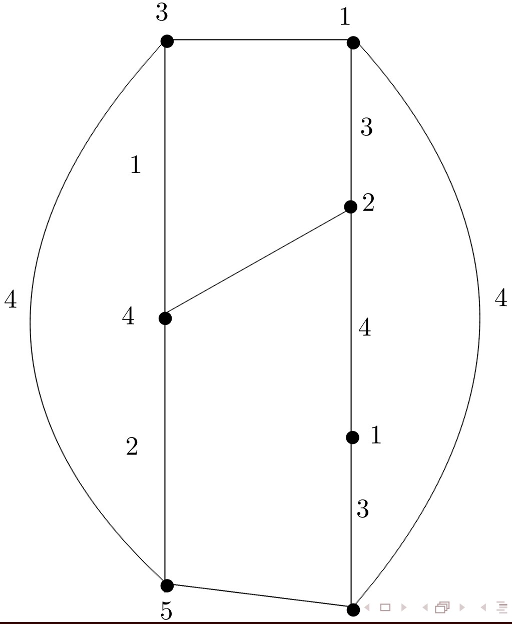
Color the vertices and edges of C_m and C_n as follows:

$$c(v_i) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

$$c(u_i) = \begin{cases} 3 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq n \\ 4 & \text{for} \quad i \equiv 0 \mod 2 \\ 5 & \text{for} \quad i = n \end{cases}$$

$$c(v_i v_{i+1}) = \begin{cases} 3 & \text{for} \quad i \equiv 1 \mod 2 \\ 4 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

$$c(u_i u_{i+1}) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq n \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \\ 4 & \text{for} \quad i = n \text{ and } i+1 = 1 \end{cases}$$


3
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2
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3
5

Proof continuation

Now, some edges between C_m and $C_n$are colored as follows.
For $1 \leq i \leq n-1$, color $c(u_iv_i) = 5$ and $c(u_nv_m) = 1$.
Then the remaining uncolored edges form a bipartite graph of maximum degree $m-1$ and it can be edge colored properly using $m-1$ new colors.

Proof continuation

Case 4. m is odd and n is even

Color the vertices and edges of both C_m and C_n using,

$$c(v_i) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq m \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \\ 3 & \text{for} \quad i = m \end{cases}$$

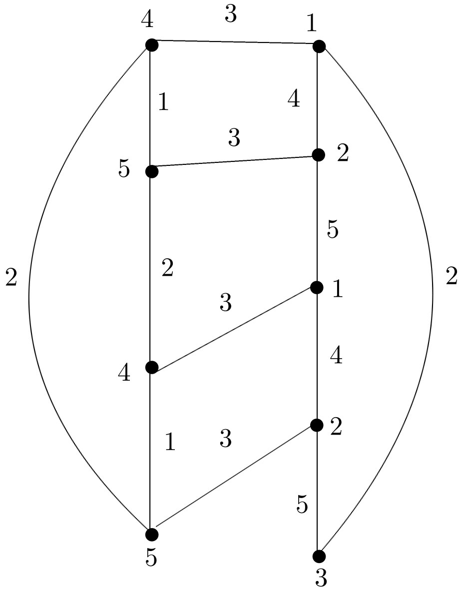
$$c(u_i) = \begin{cases} 4 & \text{for} \quad i \equiv 1 \mod 2 \\ 5 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

$$c(v_i v_{i+1}) = \begin{cases} 4 & \text{for} \quad i \equiv 1 \mod 2 \text{ and } i \neq m \\ 5 & \text{for} \quad i \equiv 0 \mod 2 \\ 2 & \text{for} \quad i = m \text{ and } i+1 = 1 \end{cases}$$

$$c(u_i u_{i+1}) = \begin{cases} 1 & \text{for} \quad i \equiv 1 \mod 2 \\ 2 & \text{for} \quad i \equiv 0 \mod 2 \end{cases}$$

Next we color the edges $\{u_i v_i : 1 \leq i \leq n\}$. For $1 \leq i \leq n$, color $c(u_i v_i) = 3$.

Remaining uncolored edges form a bipartite graph of maximum degree $m-1$ which can be edge colored properly using new $m-1$ colors.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Corollary

For any positive integers m and n, $P_m \vee P_n$ satisfies TCC.

Theorem

Let G be a graph with n vertices and $\chi'(G) = k$, Let r be the least number in $\{1, 2, \ldots, k\}$ such that every vertex of G is saturated by at least one of the matchings $M_{i_1}, M_{i_2}, \ldots, M_{i_r}$ and $\{H_1, H_2, \ldots, H_n\}$ be a set of graphs with $H_i \vee H_j$ satisfying TCC, for each

$i, j \in \{1, 2, \ldots, n\}$, then $\chi_T(G[H_1, H_2, \ldots, H_n]) \leq \sum_{i=1}^{r} s_i + \sum_{j=r+1}^{k} t_j$,

where $s_j = max\{\Delta(H_x \vee H_y) + 2 \mid xy \in M_j\}$ for $1 \leq j \leq r$ and $t_j = max\{max\{|V(H_x)|, |V(H_y)| \mid xy \in M_j\}\}$ for $r + 1 \leq j \leq k$.

Corollary

If H is any graph satisfying TCC with m vertices, then

$$\chi_T(G \circ H) \leq \begin{cases} \Delta(G \circ H) + \Delta(H)(\Delta(G) - 1) + 2\Delta(G) & \text{if } G \text{ is class} - 1 \\ \Delta(G \circ H) + \Delta(G)\Delta(H) + 2(\Delta(G) + 1) + m & \text{if } G \text{ is class} - 2 \end{cases}$$

Theorem

If G satisfies TCC with m vertices, then $G[K_n^c]$ satisfies weak TCC. In particular, if G is type-1, then $G[K_n^c]$ satisfies TCC.

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Thank You