An efficient interior point method for linear optimization using modified Newton method

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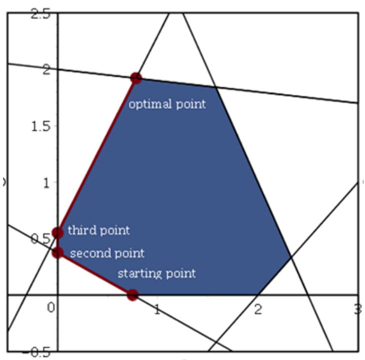

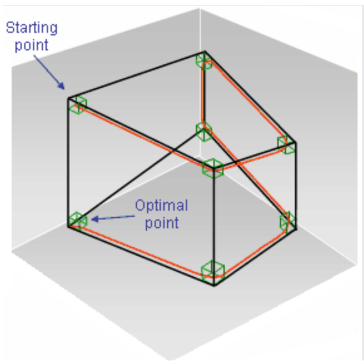

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Assumptions:

- The matrix A is full row rank, i.e., rank $(A) = m \leq n$.
- Both problems (P) and (D) satisfy the Interior Point Condition (IPC), i.e., there exists $x^0 > 0$ and (y^0, s^0) with $s^0 > 0$ such that:

$$Ax^0 = b, \qquad A^T y^0 + s^0 = c.$$

Interior point method

The KKT conditions for (P) and (D) are:

$$\begin{aligned} Ax &= b, & x &\geq 0, \\ A^T y + s &= c, & s &\geq 0, \\ xs &= 0, \end{aligned} \tag{1}$$

where the coordinate-wise product of vectors x and s is denoted as xs.

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$$\begin{aligned} Ax &= b, & x &\geq 0, \\ A^T y + s &= c, & s &\geq 0, \\ xs &= \mu \mathbf{e}, \end{aligned} \tag{2}$$

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Based on the IPC condition and the full row rank property of matrix A, the system (2) has a unique solution $(x(\mu), y(\mu), s(\mu))$. The terms $x(\mu)$ and $(y(\mu), s(\mu))$ are called the μ-centers of (P) and (D), respectively,

Solution

To find the solution, we can consider the following problem:

$$\xi = \begin{bmatrix} x \\ y \\ s \end{bmatrix} \text{ and } F(\xi) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ \mu \mathbf{e} - xs \end{bmatrix} = 0 \tag{3}$$

where the operator F is defined on the Banach space B_1 with values in a Banach space B_2. We can find the root of equation (3) denoted by ξ^* where $F(\xi^*) = 0$.

Newton's method

Applying a linear approximation using the Taylor series expansion around ξ, we have

$$F(\xi) + F'(\xi)\Delta\xi \simeq 0$$

where

$$\text{Jacobian } F' = \begin{bmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} & \frac{\partial F_1}{\partial \xi_3} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} & \frac{\partial F_2}{\partial \xi_3} \\ \frac{\partial F_3}{\partial \xi_1} & \frac{\partial F_3}{\partial \xi_2} & \frac{\partial F_3}{\partial \xi_3} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \text{ and } \Delta\xi = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix},$$

where X, S are diagonal matrices constructed from x and s.

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where X, S are diagonal matrices constructed from x and s.
We update the current estimate ξ_n of the root using the following rule for some appropriate step size α:

$$\xi_{n+1} = \xi_n - \alpha[F'(\xi_n)]^{-1}F(\xi_n)$$

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First step:

$$\begin{aligned}\tilde{x}_0 &= x_0\\ x_1 &= x_0 - \frac{f(x_0)}{f'(\frac{1}{2}[x_0 + \tilde{x}_0])} = x_0 - \frac{f(x_0)}{f'(x_0)},\end{aligned}$$

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and for $k \geq 1$, we have:

Second step:

$$\begin{aligned} \tilde{x}_k &= x_k - \frac{f(x_k)}{f'(\frac{1}{2}[x_{k-1} + \tilde{x}_{k-1}])} \\ x_{k+1} &= x_k - \frac{f(x_k)}{f'(\frac{1}{2}[x_k + \tilde{x}_k])}. \end{aligned}$$

Two-step method for IPM

Let F be any function. The first step involves updating an auxiliary point $\tilde{\xi}_0 = \xi_0$.
The update rules used in the n^{th} iteration can be concisely summarized as:

$$
\begin{aligned}
\tilde{\xi}_{n+1} &= \xi_n - \alpha[F'(\hat{\xi}_n)]^{-1}F(\xi_n) \\
\hat{\xi}_{n+1} &= \frac{1}{2}(\tilde{\xi}_{n+1} + \xi_n) \\
\xi_{n+1} &= \xi_n - \alpha[F'(\hat{\xi}_{n+1})]^{-1}F(\xi_n)
\end{aligned}
$$

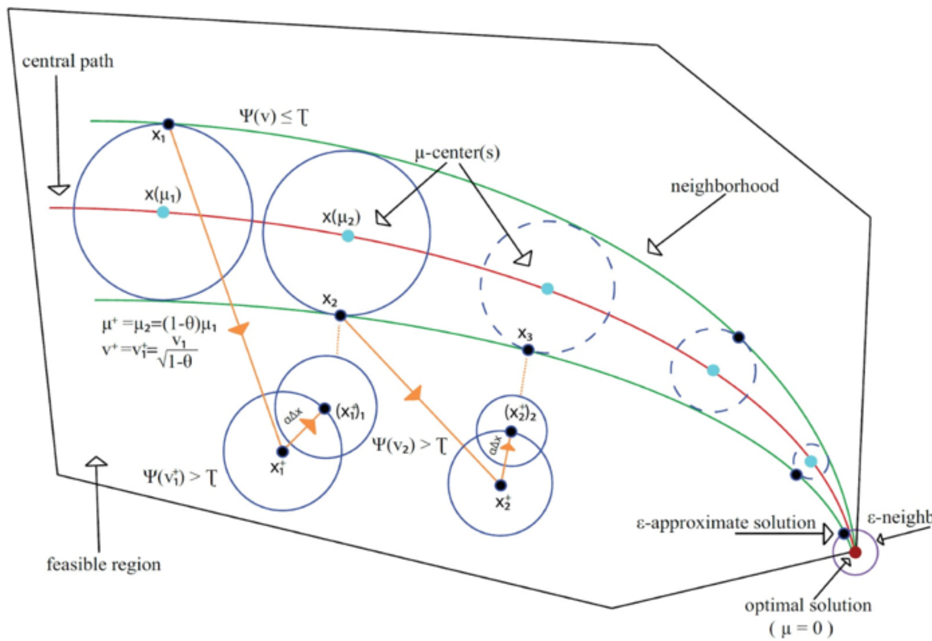
Interior point algorithm

Algorithm 1 Generic Primal–dual IPM for LO.

Input

a proximity function $\Psi(v)$
a threshold parameter $\tau > 0$
an accuracy parameter $\varepsilon > 0$
a barrier update parameter $\theta, 0 < \theta < 1$

begin
  $x := \mathbf{e}; s := \mathbf{e}; \mu := 1; v := \mathbf{e};$
  while $n\mu > \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu;$
    while $\Psi(v) > \tau$ do
    begin
      $x := x + \alpha \Delta x$
      $s := s + \alpha \Delta s$
      $y = y + \alpha \Delta y$
      $v := \sqrt{\frac{xs}{\mu}}$
    end
  end
end



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Check the condition $\Psi \geq \tau$. If $\Psi < \tau$ go back to step 2.

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- Set $(\tilde{x}, \tilde{y}, \tilde{s}) = (x, y, s)$.
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$$\begin{aligned} A\Delta x &= 0 \\ A^T\Delta y + \Delta s &= 0 \\ \hat{S}\Delta x + \hat{X}\Delta s &= \mu\mathbf{e} - XS\mathbf{e} \end{aligned} \tag{4}$$

Note the $\hat{X}, \hat{S}$ are diagonal matrices constructed from $\hat{x}, \hat{s}$.

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$$(x_+, y_+, s_+) \leftarrow (x + \beta\Delta x, y + \beta\Delta y, s + \beta\Delta s) \tag{5}$$

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- Check Step 1. If $\Psi(x, s, \mu) < \tau$ stop the inner loop.

Algorithm

Algorithm 2: A two-step feasible IPM algorithm for LPs.

Input: $x_0,\ y_0,\ s_0,\ \mu > 0,\ \tau > 0,\ \varepsilon > 0,\ \theta \in (0,1)$, and $\Psi(x, s, \mu)$

$(x, y, s) \leftarrow (x_0, y_0, s_0)$

$k, m \leftarrow 0$

while *stopping criteria is not met* **do**

  $\mu \leftarrow \mu(1-\theta)$

  while $\Psi(x, y, s, \mu) \geq \tau$ **do**

    if $m{=}{=}0$ **then**

      $(\tilde{x}, \tilde{y}, \tilde{s}) \leftarrow (x, y, s)$

      Update $(\hat{x}_+, \hat{y}_+, \hat{s}_+) \leftarrow \frac{1}{2}((\tilde{x}, \tilde{y}, \tilde{s}) + (x, y, s))$

      Find search direction $(\Delta\tilde{x}, \Delta\tilde{y}, \Delta\tilde{s})$ using (6)

      Find the value for β and update $(x, y, s) \leftarrow (x + \beta\Delta x, y + \beta\Delta y, s + \beta\Delta s)$

    if $m \geq 1$ **then**

      Find the search direction using (6)

      Find the maximum step size α and update the auxiliary point using (7)

      Update the average point by using (8)

      Find search direction $(\Delta x, \Delta y, \Delta s)$ by solving (9)

      Find the maximum value for step size β and update the current point by using (10)

    $m \leftarrow m + 1$

  $k \leftarrow k + 1$

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Lemma:

The total number of outer iterations to obtain $n\mu \leq \epsilon$ are

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Theorem:

Suppose the outer loop updates the barrier parameter by factor $\theta \in (0,1)$ and $k \to \infty$. Then one has:

$$\|\xi_k - \xi^*\| \leq \epsilon.$$

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Stopping condition:

For both algorithms, we stopped if the number of iterations exceeded 700 or if the relative gap was less than 10^{-6}. The relative gap is the absolute difference between $c^T x$ and $b^T y$ divided by $1 + |c^T x| + |b^T y|$.

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We have selected 46 test problems of varying sizes from the Netlib collection.

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As for the barrier parameter, we have set the initial value to $\mu_0 = 1$ for both algorithms. In each iteration of the outer loop of the algorithm, we reduce the value of μ by $\mu = (1 - \theta)\mu$.

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Proximity function:

For our algorithms, we rely on the proximity function specified as follows: $\Psi(x, s, \mu) = \|\mu\mathbf{e} - xs\|_1$, where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

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We are currently working on a large-update method, τ value should be set to $O(n)$.

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Step size:

We use the following equations:

$$\alpha_x^{\max} = \frac{1}{\max_{i=1,2,\cdots,n}\{1, -\frac{x_i}{\Delta x_i}\}}, \qquad \alpha_s^{\max} = \frac{1}{\max_{i=1,2,\cdots,n}\{1, -\frac{s_i}{\Delta s_i}\}}.$$

To ensure we don't hit the boundary, we reduce the maximum allowable step sizes by a fixed factor of $0 < \alpha_0 < 1$. Therefore, our final step sizes are given by $\alpha_x = \alpha_0.\alpha_x^{\max}$ and $\alpha_s = \alpha_0.\alpha_s^{\max}$.

Results

Methods	Aver. Iter.	Aver. CPU
Classical Algorithm	95.29	132.46
Algorithm 2	**65.77**	**105.43**

Table 1: The average number of iterations and CPU time

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- The new proposed approach can significantly reduce the number of iterations and CPU times by %30.97 and %20.46, respectively.

References

- [1] Roos, C., Terlaky, T., Vial, J.P.: Theory and algorithms for linear optimization: an interior point approach. Wiley Chichester (1997).
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Thank You For Your Attention!

Any Questions?