Generalized Minimum-Membership Geometric Set Covering

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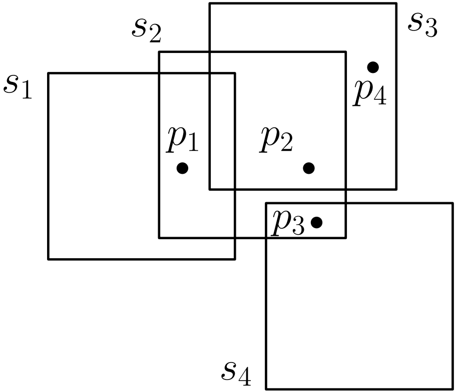
Part 1:

Problem Definition and History

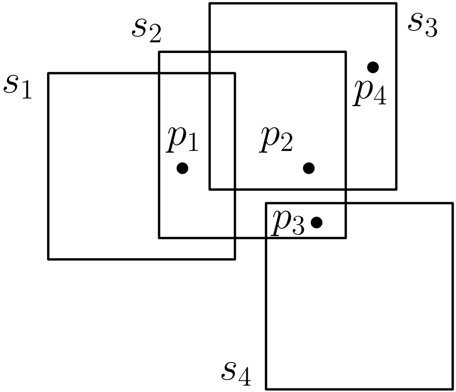
Geometric Set Cover



s_1 **covers** p_1.

s_1 **does not cover** p_2.

Geometric Set Cover



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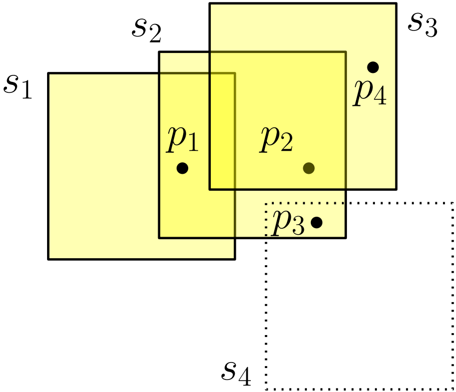
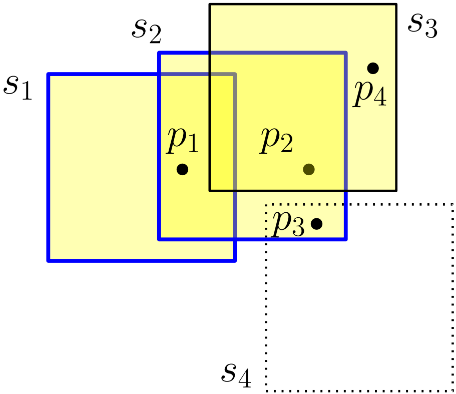


s_1 **covers** p_1.

s_1 **does not cover** p_2.

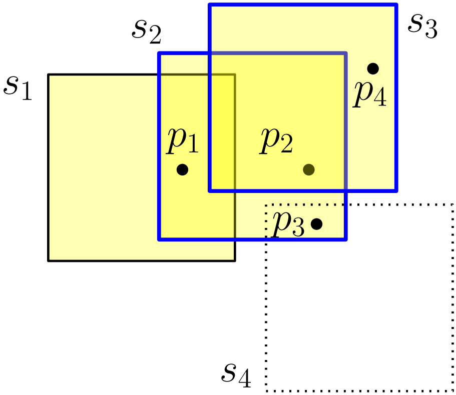
$\{s_1, s_2, s_3\}$ is a **cover** for $\{p_1, p_2, p_3, p_4\}$.

Membership of an input point w.r.t. a Cover



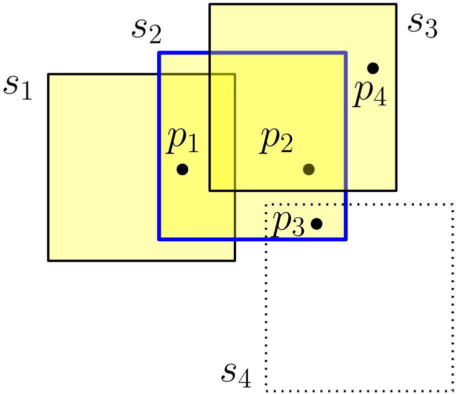
membership$(p_1) = 2$.

Membership of an input point w.r.t. a Cover



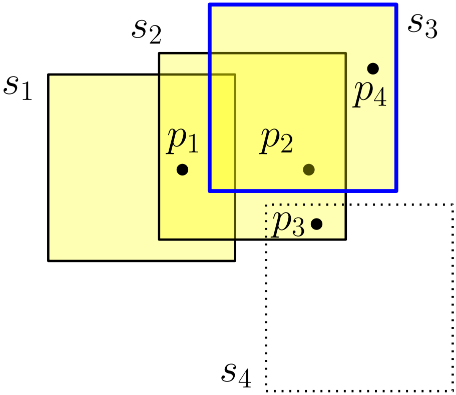
membership(p_2) $= 2$.

Membership of an input point w.r.t. a Cover



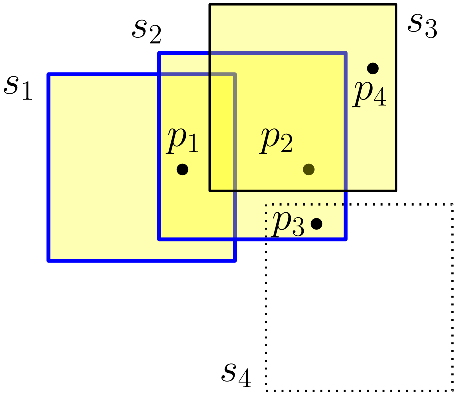
membership(p_3) $= 1$.

Membership of an input point w.r.t. a Cover



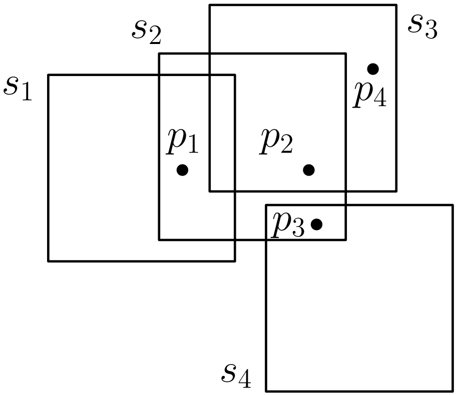
membership(p_4) $= 1$.

Membership of a point set w.r.t. a Cover



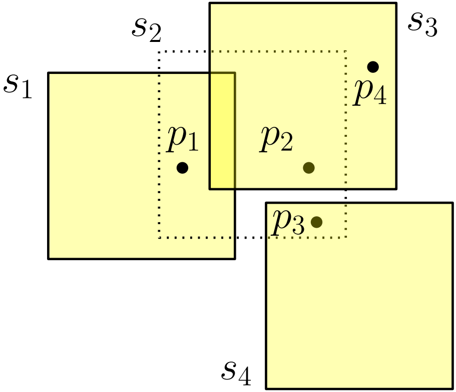
$$\text{membership}(\{p_1, p_2, p_3, p_4\}) = 2.$$

Minimum Membership Cover



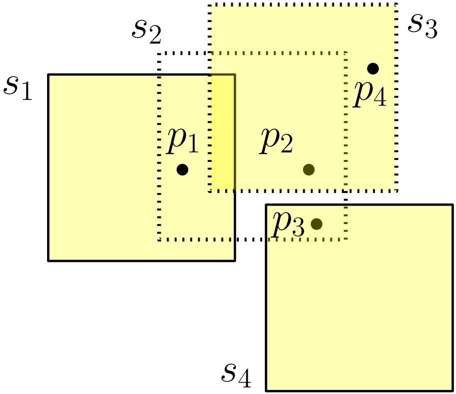
Cover all the points such that membership is minimized.

Minimum Membership Cover



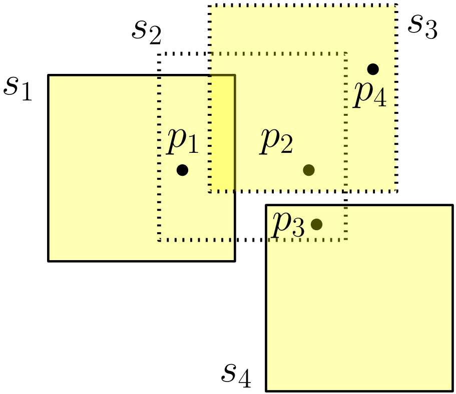
cardinality $= 3$, membership $= 1$.

Minimum Membership Cover vs Minimum Cardinalty Cover



A minimum cardinality cover is $\{s_2, s_3\}$. (dotted squares)

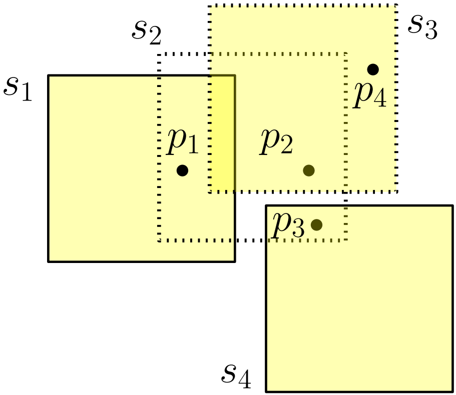
Minimum Membership Cover vs Minimum Cardinalty Cover



A minimum cardinality cover is $\{s_2, s_3\}$. (dotted squares)

Membership w.r.t. this minimum cardinality cover is 2.

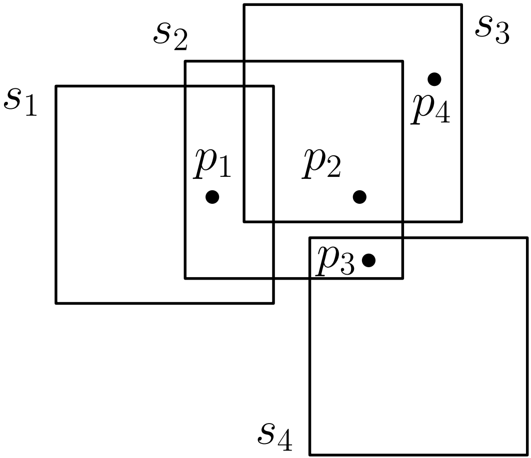
History of Minimum Membership Cover



Minimum **Membership** Cover [SODA 2008]

- Erlebach and van Leeuwen show that the problem is NP-hard.
- Gave a poly-time 5-**approx** for unit squares, when there exists a cover with **constant membership**.

Ply of a set of objects



Ply is the maximum number of overlapping objects.

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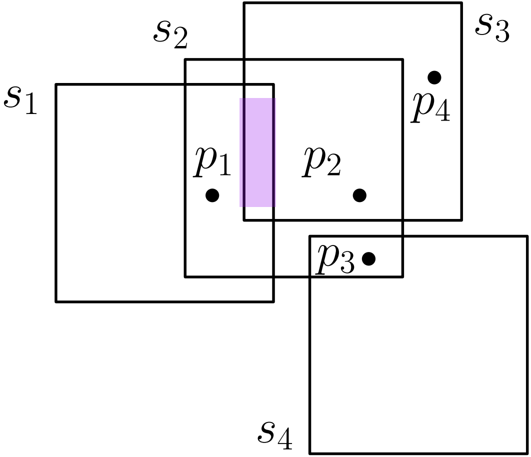
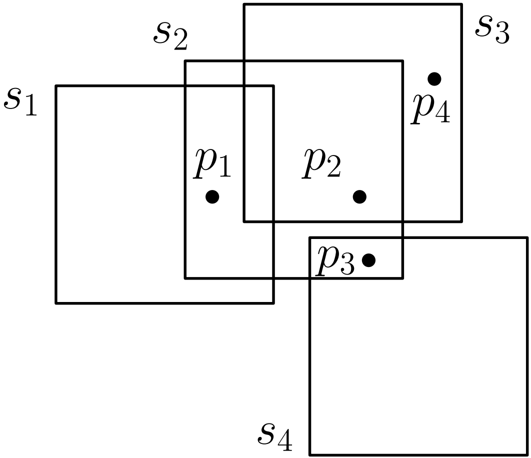


$$\text{ply}(\{s_1, s_2, s_3, s_4\}) = 3$$

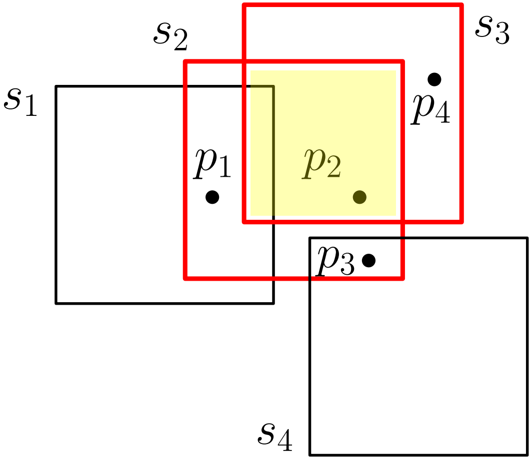
Ply is the maximum number of overlapping objects.

Minimum Ply Cover



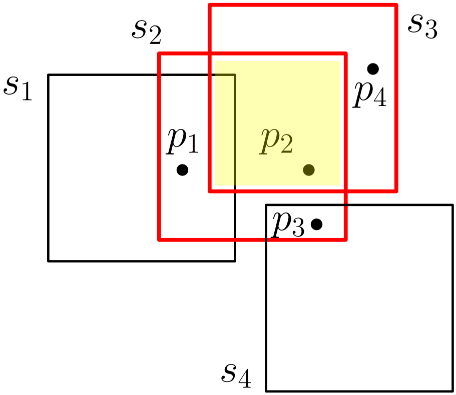
Cover all the points such that ply is minimized.

Minimum Ply Cover



ply $= 2$.

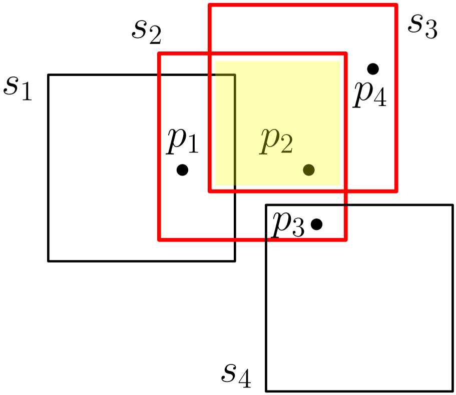
History of Minimum Ply Cover



Minimum Ply Cover [Computational Geometry 2021]

- Biedl, Biniaz and Lubiw proved that it is NP-hard.
- Gave a poly-time 2-**approx** for unit squares and unit disks.
- Assumption: there exists a cover for the instance with **constant ply**.

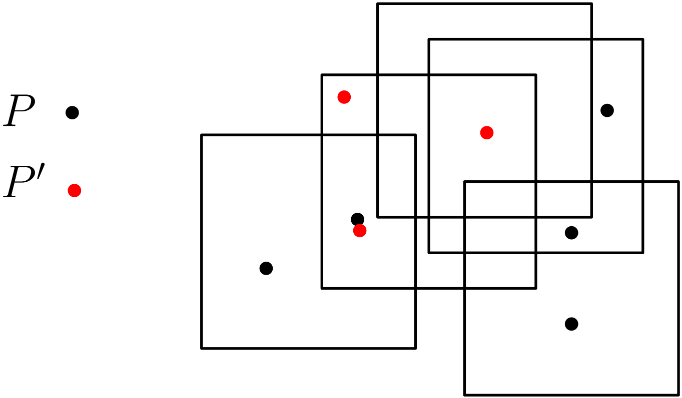
History of Minimum Ply Cover



Minimum Ply Cover [WALCOM 2023]

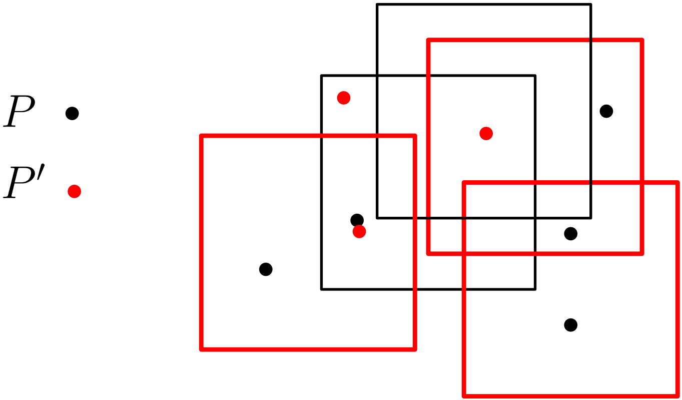
- Durocher, Keil and Mondal gave an 8-approx for unit squares.
- The first poly-time constant approximation.
- No assumptions on the minimum **ply** value.

Generalized Minimum Membership Cover



Cover the **black points** such that membership of the red point set is **minimized**.

Generalized Cover

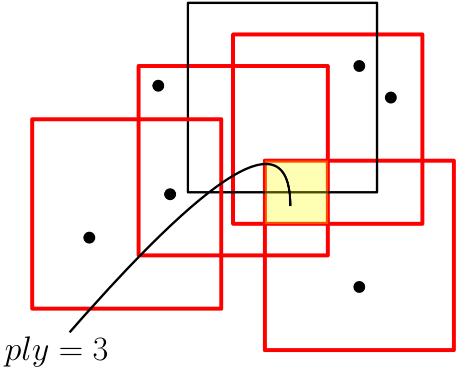


membership of the red point set is 1 (wrt. the solution).

Generalized Cover → the two Special Cases

Membership and Ply as special cases

- $P' = P \implies$ Minimum Membership Cover.
- $P' \equiv \mathbb{R}^2 \implies$ Minimum Ply Cover.



membership $= 2$, ply $= 3$.

History of Generalized Cover

Generalized Cover [SoCG 2023]

- Bandyapadhyay, Lochet, Saurabh, Xue gave an 144-approx for unit squares.
- The first polynomial-time constant approximation.
- Applies to both the ply and membership versions.
- No assumptions on the minimum **membership** value.
- **Runs in** $\tilde{O}(mn)$. [$m =$ no. of unit squares, $n =$ no. of points]

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Part 2:

Our Results and Proof Sketches

Our Results

Generalized Cover [This work]

- We gave a 16-approx for unit squares.
- No assumptions on the minimum **membership** value.
- **Runs in** $O(m^2 \log m + m^2 n)$. [m = no. of unit squares, n = no. of points]

Improvement in Approximation Ratio:

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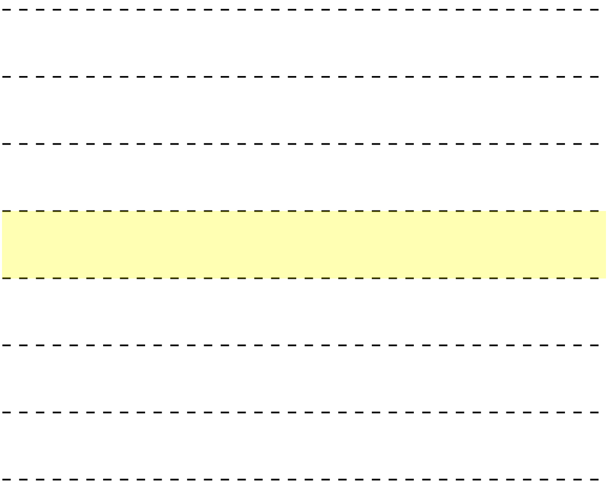
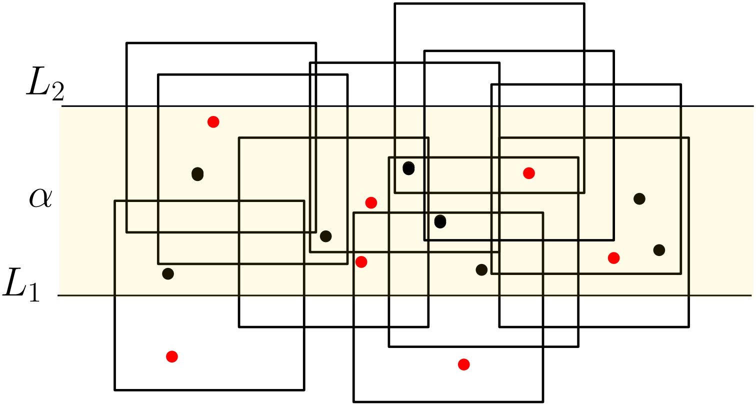
Partitioning the input into subproblems



Slab decomposition

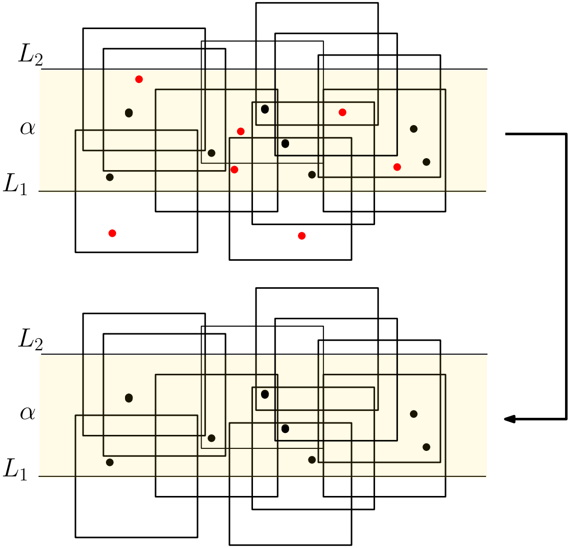
Zooming into one Slab subproblem



All squares intersect the slab.

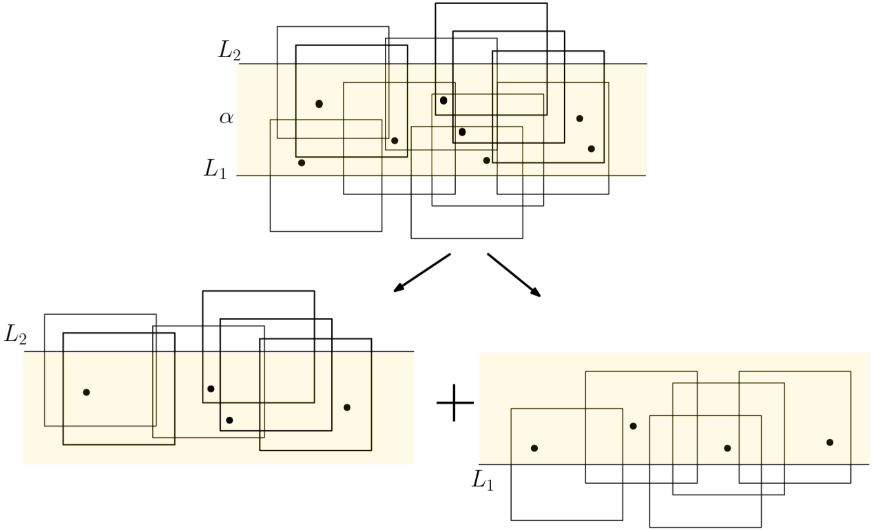
All the **black points** are within the slab.

Partitioning the Slab subproblem into Line subproblems



Partition the squares and **black points** only.

Slab Decomposed into Line Subproblems



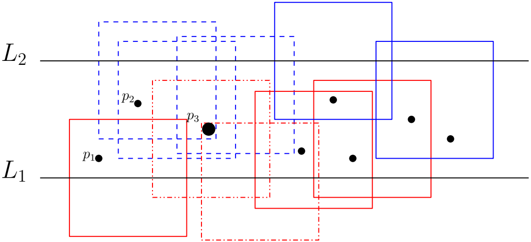
1. The squares are naturally partitioned.
2. For partitioning the black points, use an LP technique.

Partioning the points: LP technique

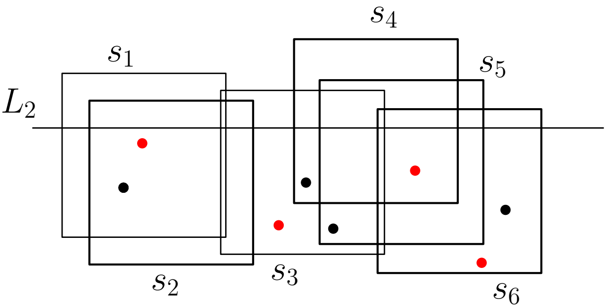
[From Bandyapadhyay, Lochet, Saurabh, Xue (SoCG 2023)]

- $\forall s \in \mathcal{S}$, take a variable x_s. Take y for membership.

$$\begin{aligned} & \min y \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}, p \in s} x_s \geq 1, && \text{for all } p \in P \\ & \sum_{s \in \mathcal{S}, p' \in s} x_s \leq y, && \text{for all } p' \in P' \\ & 0 \leq x_s \leq 1, && \text{for all } s \in \mathcal{S} \end{aligned}$$

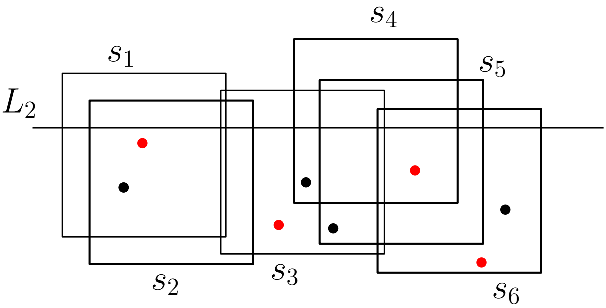


The Line subproblem



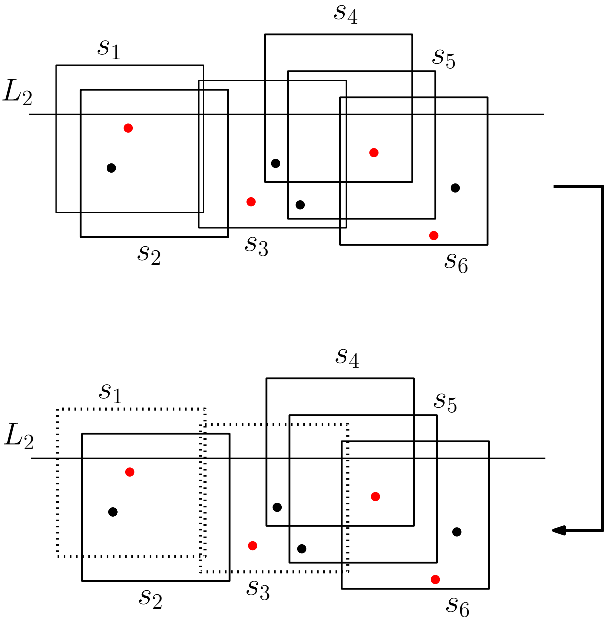
- All squares intersect a horizontal line.
- All black points lie below the horizontal line.

Running the algorithm on the Line subproblem

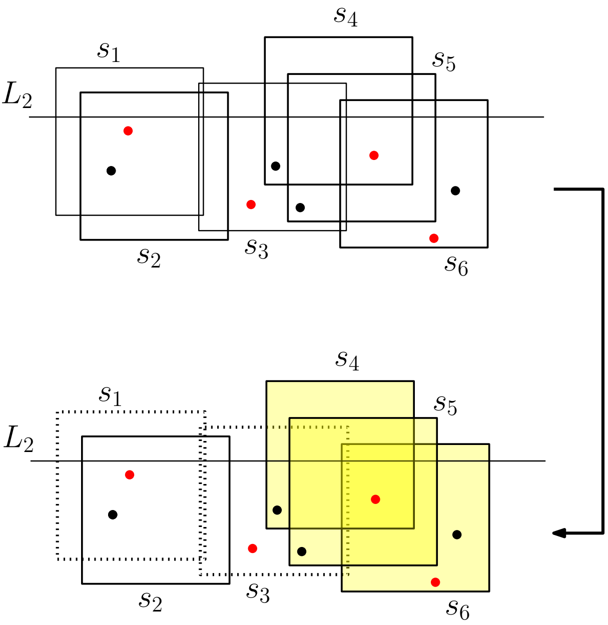


- Running the Algorithm ...

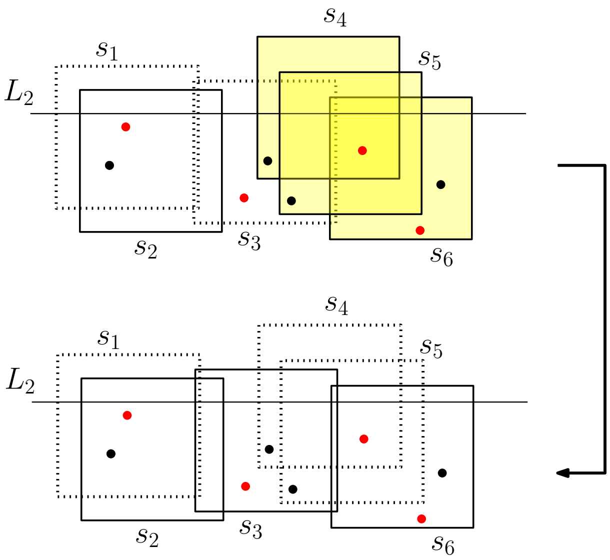
Step 1: Remove Redundancy



Step 2: Identify the maximum discrete clique



Step 3: Execute profitable swaps, if any



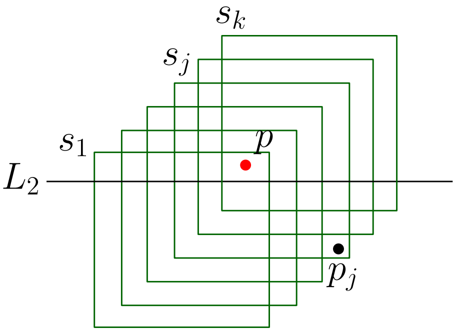
Swap in s_3 and swap out s_4, s_5.

Algorithm for a Line Subproblem

Algorithm Sketch

1. Remove redundant squares from the input.
2. Identify a maximum discrete clique, say Q.
3. **While** there is a profitable swap in Q **do**
   a. Reduce size of Q by performing profitable swaps.
   b. Remove redundant squares, if any.
4. Return the solution.

Proof Idea



The maximum membership is realized at the red point p.

Algorithm outputs a solution with membership k.

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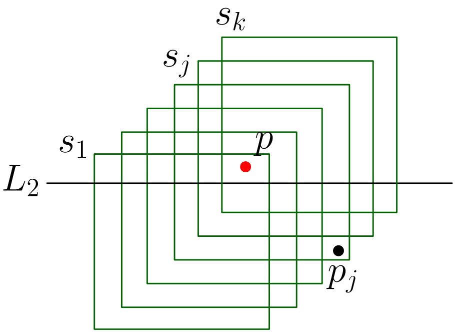


The maximum membership is realized at the red point p.

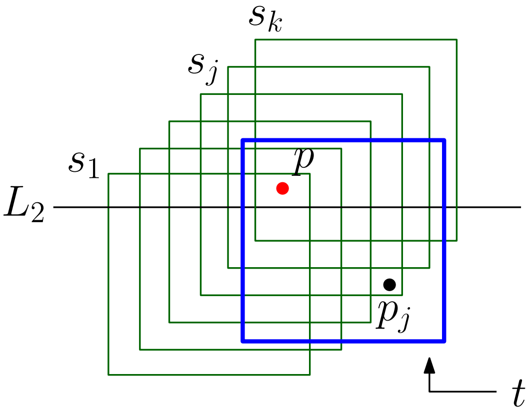
Algorithm outputs a solution with membership k.

Goal: Show that $k/4$ is a lower bound on OPT.

Observations about the structure of our solution

Property 1

Containing $p_j \implies$ containing p (for each input square).

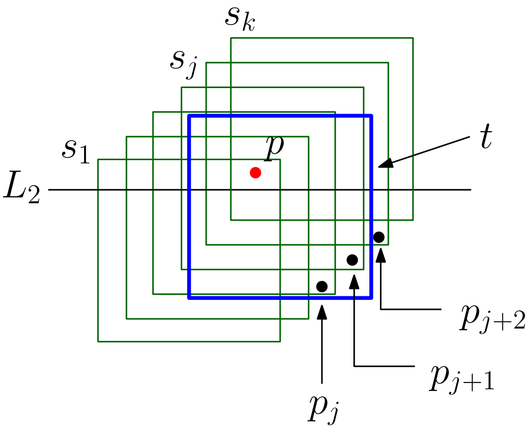


p_j is the **bottom-most exclusive** point of s_j.

Observations about the structure of our solution

Property 2

No input square contains p_j, p_{j+1}, p_{j+2} simultaneously.



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- At least $k - 9$ squares will obey both the properties 1 and 2.
- Let J denote the index set of such squares.

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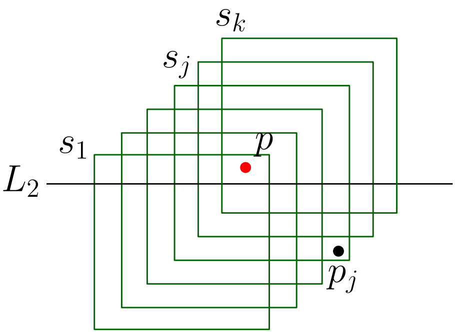
Analysis of 4-approximation



$$OPT \geq \sum_{p \in s} x_s$$

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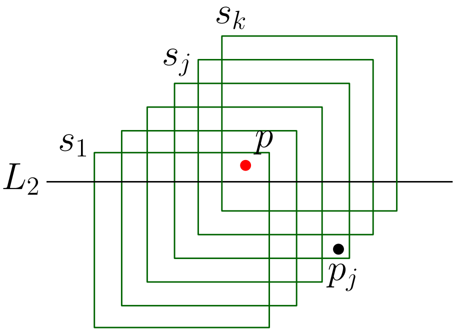
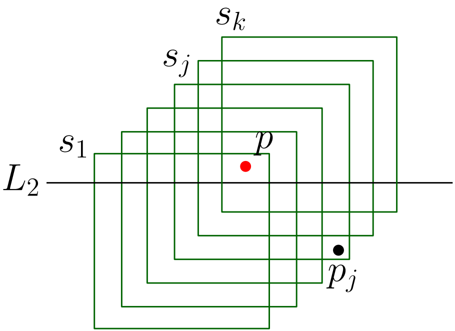


$$OPT \geq \sum_{p \in s} x_s \geq \frac{1}{2} \sum_{\forall j \in J} \sum_{p_j \in s} x_s$$

The set of squares containing p_j is a subset of the set of squares containing p.

Analysis of 4-approximation



$$OPT \geq \sum_{p \in s} x_s \geq \frac{1}{2} \sum_{\forall j \in J} \sum_{p_j \in s} x_s \geq \frac{1}{2} \cdot (k-9) \cdot \frac{1}{2}$$

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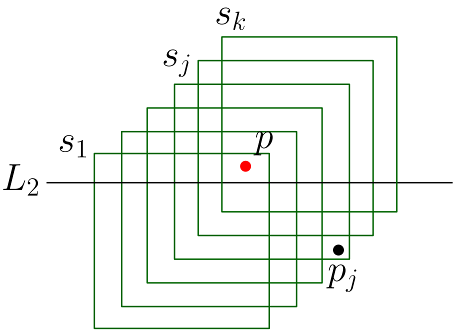

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$$\implies k \leq 4 \cdot OPT + 9$$

Main Theorem

Theorem

GMMGSC problem admits an algorithm that runs in $O(m^2 \log m + m^2 n)$ time, and computes a set cover whose membership is at most $16 \cdot OPT + 36$, where OPT denotes the minimum membership.

Summary of the Results

Paper	**Generalized**	**Ply only**	**Running Time**
Durocher et al.	NA	8	$O((n+m)^{12})$
Bandyapadhyay et al.	144	144	$\tilde{O}(nm)$
	NA	36	$n^{O(1/\epsilon^2)}$
Our Paper	16	16	$O(m^2 \log m + m^2 n)$

[$m =$ no. of unit squares, $n =$ no. of points]

Thank you