On Total Chromatic Number of Complete Multipartite Graphs

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Overview

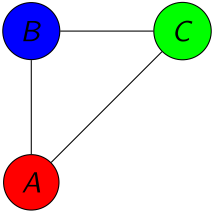
Definitions and Background

Vertex Coloring

Vertex Coloring

A proper vertex coloring of a graph G is an assignment of colors to the vertices such that no two adjacent vertices receive the same color.

The Chromatic Number of G, denoted by $\chi(G)$, is the minimum number of colors required for a proper vertex coloring of a graph G.



$$\chi(K_3) = 3$$

Vertex Coloring

Theorem 1 (Brooks 1941)

For a graph G with maximum degree $\Delta(G)$,

$$\chi(G) \leq \begin{cases} \Delta(G)+1 & \textit{if } G \textit{ is a complete graph or an odd cycle,} \\ \Delta(G) & \textit{otherwise.} \end{cases}$$

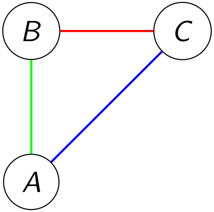
Edge Coloring of Graphs

Edge Coloring

Edge Coloring

A proper Edge Coloring of a graph G is an assignment of colors to its edges such that no two edges incident on a vertex receive the same color.

The Edge Chromatic Number of G, denoted by $\chi'(G)$, is the minimum number of colors required for a proper edge coloring of the graph G.



$$\chi'(K_3) = 3$$

Edge Coloring

Theorem 2 (Vizing 1964)

For a graph G with maximum degree $\Delta(G)$,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

Definition 3

G is of Class 1 if $\chi'(G) = \Delta$ and is of Class 2 if $\chi'(G) = \Delta + 1$

Total Coloring

Total Coloring

A total coloring of a graph G is a coloring of the vertices and the edges such that:

- no two adjacent vertices receive the same color (vertex coloring),
- no two adjacent edges receive the same color (edge coloring),
- **If a vertex x is incident on an edge e (end point of e), then x and e receive different colors .**

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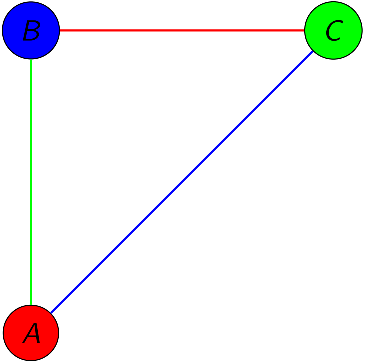
The Total Chromatic Number of G, denoted by $\chi_T(G)$ or $\chi''(G)$, is the minimum number of colors required for a total coloring a graph G.

Total Coloring



$$\chi''(K_3) = 3$$

Total Coloring

Total Coloring Conjecture (Behzad 1965, Vizing 65)

Let $\Delta(G)$ is the maximum degree of the graph G.

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$$

Total Coloring Classification Problem

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For a graph G which satisfies the Total Coloring Conjecture, there is a natural classification into two categories:

$$G \text{ is of } \begin{cases} \text{Type 1} & \text{if } \chi'' = \Delta(G) + 1, \\ \text{Type 2} & \text{if } \chi'' = \Delta(G) + 2. \end{cases}$$

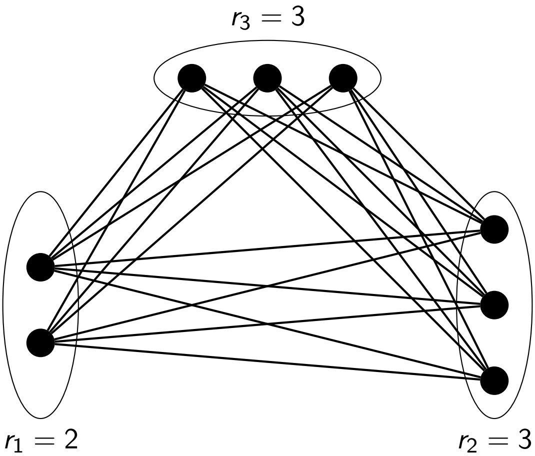
Complete Multipartite Graphs

Definition 4

The complete p-partite graph $K = K(V_1, V_2, \dots, V_p)$, $|V_i| = r_i$, $r_1 \leq r_2, \cdot, \leq r_p$, is the graph with vertex set $V(K) = \cup_{i=1}^{p} V_i$ (each set V_i is called a part) in which two vertices are joined if and only if they occur in different parts of K. It is also denoted as $K = K(r_1, r_2, \dots, r_p)$

Complete Multipartite Graphs

$K(r_1, r_2, r_3) = K(2, 3, 3)$



Complete Multipartite graph is simple but Complex

- Complete Multipartite graph is sufficiently complex that settling the values of its graph parameter is often a challenge.
- It was only in 1992, the classification problem for this class with respect to the chromatic index was settled.
- It was shown by Hoffman and Rodger that K is of class 2 ($\chi'(G) = \Delta(G) + 1$) iff it is overful ($|E| > \Delta(G)\lfloor \frac{n}{2} \rfloor$)

TCC for Complete Multipartite Graphs

Theorem 5 (Rosenfeld (71))

For a balanced complete multipartite graph $K(V_1, V_2, \ldots, V_p)$ (all parts are of same size),

$$\chi''(K) \leq \Delta(K) + 2$$

Lemma 6 (Chetwyand and Hilton; Yap, Wang and Zhang [1986])

Let G be a graph of order n. If G contains an independent set S of vertices such that $|G - S| \leq \Delta + 1$, then

$$\chi''(G) \leq \Delta + 2$$

Theorem 7 (Yap[89])

For a complete multipartite graph $K(V_1, V_2, \ldots, V_p)$,

$$\chi''(K) \leq \Delta(K) + 2$$

Proof.

Take $S = V_p$, the largest part. Now $|V(K) \setminus S| \leq \Delta(G) + 1$. Hence TCC holds for $K(V_1, V_2, \ldots, V_p)$. □

Classification: Complete Multipartite graph with odd Order

Theorem 8 (Chew and Yap[92]; Hoffman and Rodger[92])

Let K be a complete multipartite graph such that $|V(K)|$ is odd,

$$\chi''(K) = \Delta(K) + 1$$

Balanced complete Multipartite Graph

Theorem 9

(Bermond (1974) $\chi_T(K(r,n)) = \Delta(K(r,n)) + 2$ *if* $r = 2$ *or* r *is even and* n *is odd else* $\Delta(K(r,n)) + 1$
($K(r,n)$ *denotes balanced complete multipartite graph having* r *parts each of size* n*)*

Classification: Complete Multipartite graph with even Order

Theorem 10 (Hoffman and Rodger(96))

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)| = 2n$. If

$$def(K) \geq \begin{cases} 2n - r_1 & \text{if } p = 2 \text{ or} \\ & \text{if } p \text{ is even, } r_1 \text{is odd, and } r_1 = r_{p-1}, \\ 2n - r_p & \text{otherwise,} \end{cases}$$

then K is of Type 1, where $def(G) = \Sigma_{v \in V(G)}(\Delta(G) - d_G(v))$.

Conjecture

Conjecture (Hoffman and Rodger(96))

A complete multipartite graph $K(r_1, r_2, \ldots, r_p)$ is of Type 2 if and only if

1. $p = 2$ and K is regular, or
2. $|V(K)|$ is even and $def(K)$ is less than the number of parts of odd size.

Results based on size of parts

Theorem 11 (Chew and Yap[92])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph. If $r_1 < r_2$, then K is of Type 1.

Theorem 12 (Dong and Yap[2000])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)|$ is even. If $r_2 \leq r_3 - 2$, then K is of Type 1.

Theorem 13 (Dalal, Panda and Rodger[2016])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)|$ is even. If $r_2 < r_3$, then K is of Type 1.

Theorem 14 (Dalal, Panda and Rodger[2023])

Let $K(r_1, r_2, \ldots, r_p)$ be a complete multipartite graph such that $|V(K)|$ is even. If $r_3 < r_4$, then K is of Type 1.

Results based on numbers of parts

Theorem 15 (Chew and Yap[92])

Let $K(r_1, r_2, r_3)$ be a complete 3-partite graph of even order. Then K is of Type 1.

Theorem 16 (Dong and Yap[2000])

Let $K(r_1, r_2, \ldots, r_4)$ be a complete 4-partite graph of even order. Then K is of Type 2 if and only if def(K) is less than the number of parts of odd size.

Theorem 17 (Dalal and Rodger[2014])

Let $K(r_1, r_2, \ldots, r_5)$ be a complete 5-partite graph of even order. Then K is of Type 2 if and only if def(K) is less than the number of parts of odd size.

Techniques Used in Proofs

- Traditional Proof Technique (β-Biased Coloring)
- Amalgamation technique [Dalal and Rodger (2014)]

β-biased coloring

3 Steps:

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- Color the vertices of one part, say β, using one color. Color the maximum matching M in the remaining graph $G - \beta$ with the same color

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- Add a vertex v to all the vertices of the graph, except for vertices in β.

β-biased coloring

3 Steps:

- Color the vertices of one part, say β, using one color. Color the maximum matching M in the remaining graph $G - \beta$ with the same color
- Add a vertex v to all the vertices of the graph, except for vertices in β.
- Using the edge coloring techniques, find the proper edge coloring of the $G^* = G + \{v\} - M$ using $\Delta(G^*)$.

Limitations

- Edge coloring classification problem itself is open.
- Most of the results use the property that if a core of a graph is a forest, then it is class 1. (core of a graph is the sub-graph induced by the vertices of maximum degree). However, not many graphs can have this property. For example, $K(r, r, r, r, r, r, r+1)$

Graph Amalgamations

- H is an *amalgamation* of G if there exists a function ψ called an amalgamation function from $V(G)$ onto $V(H)$ and a bijection $\phi' : E(G) \to E(H)$ such that e joins u and v in $E(G)$ if and only if $\phi'(e)$ joins $\psi(u)$ and $\psi(v)$ in $E(H)$.

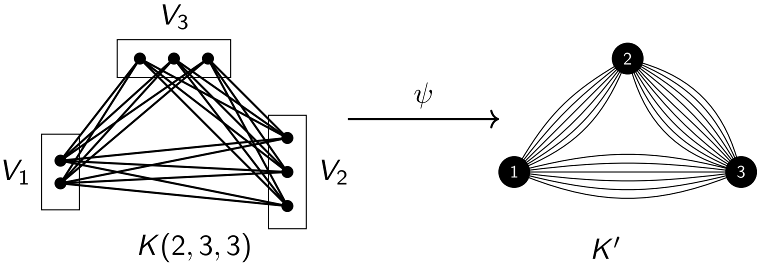


Figure: Amalgamation of $K(2,3,3)$ into K' where $V(K') = \{1,2,3\}$.

Graph Amalgamations

- Associated with ψ is the *number function* $\eta : V(H) \to \mathbb{N}$ defined by $\eta(v) = |\psi^{-1}(v)|$, for each $v \in V(H)$. G is a *detachment* of H if there exists an amalgamation function ψ of G onto H such that $|\psi^{-1}(\{u\})| = \eta(u)$ for every $u \in V(H)$. Some authors refer to detachments as *disentanglements*.
- The subgraph of G induced by the edges colored j is denoted by $G(j)$. For a graph G, $m_G(u, v)$ denotes the number of edges joining vertices u and v in G, and $l_G(u)$ denotes the number of loops incident to vertex u. If x, y are real numbers, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integers such that $x - 1 \leq \lfloor x \rfloor \leq x \leq \lceil x \rceil \leq x + 1$, and $x \approx y$ means $\lfloor y \rfloor \leq x \leq \lceil y \rceil$.

Bahamanian and Rodger [J. Graph Theory (2012)]

Theorem 18

Let H be a k-edge-colored graph and η be a function from $V(H)$ into $\mathbb{N}$ such that for each $w \in V(H)$, $\eta(w) = 1$ implies $l_H = 0$. Then there exists a loopless η-detachment G of H with amalgamation function $\psi : V(G) \longrightarrow V(H)$, η being the number function associated with ψ such that G satisfies the following conditions:

1. $d_G(u) \approx d_H(w)/\eta(w)$ *for each* $w \in V(H)$ *and each* $u \in \psi^{-1}(w)$;
2. $d_{G(j)}(u) \approx d_{H(j)}(w)/\eta(w)$ *for each* $w \in V(H)$, *each* $u \in \psi^{-1}(w)$ *and each* $j \in \mathbb{N}_k$;
3. $m_G(u, v) \approx m_H(w, z)/(\eta(w)\eta(z))$ *for every pair of distinct vertices* $w, z \in V(H)$, *each* $u \in \psi^{-1}(w)$ *and* $v \in \psi^{-1}(z)$; *and*
4. $m_{G(j)}(u, v) \approx m_{H(j)}(w, z)/(\eta(w)\eta(z))$ *for every pair of distinct vertices* $w, z \in V(H)$, *each* $u \in \psi^{-1}(w)$, $v \in \psi^{-1}(z)$ *and each* $j \in \mathbb{N}_k$.

Total Coloring using Graph Amalgamations

Our technique

- Let K' be an amalgamation of complete multipartite graph K, where $|V(K')| = k'$, ψ and η be the *amalgamation function* and the associated *number function*, respectively. For simplicity, assume that $V(K') = \mathbb{N}_{k'}$.
- Find $\Delta(K) + 1$ subgraphs of K', each with degree sequence majorized by $(\eta(1)), \ldots, \eta(k'))$, so that their union forms a graph in which for $1 \leq i < j \leq \eta(k')$ vertices i and j are joined by exactly $\eta(i)\eta(j)$ or 0 edges as the case may be,
- Disentangle K' to get K (using Bahamanian-Rodger Theorem) , the vertices of K are colored using only the $\Delta(K) + 1$ colors while ensuring that the properties of total coloring are satisfied.

It is convenient to consider each subgraph as a color class.

Total Coloring using Graph Amalgagmations

Our technique follows four general steps:

1. k' or $k'-1$ color classes of K' are defined in which each vertex $i \in \{1, \ldots, k'\}$ has degree as close to $\eta(i)$ as possible except for one which has degree 0.
2. Many color classes of K' with degree sequence $(\eta(1)), \ldots, \eta(k'))$ saturating all the vertices are found.
3. The remaining edges of K' are partitioned into few remaining color classes.
4. Using Bahamanian-Rodger Theorem , K' is disentangled to get K. Most vertices of K are then colored using colors in step (1) and left ones are colored using color(s) in step (3).

This may be seen in contrast with the technique of Dalal and Rodger [Graphs and Combinatorics (2014)], though not stated in above form.

Dalal and Rodger [Graphs and Combinatorics (2014)]

- For a complete multipartite $K = K(V_1, \ldots, V_p)$, their approach would always have a fixed *amalgamation function* $\psi : V(K) \to V(K')$ such that $\psi(V_i) = \{i\}$ and $\eta(i) = r_i$ for $1 \leq i \leq p$.
- In Step (1), p color classes would be defined.
- In Step (4), the following result of Leach and Rodger is used, and only p color classes (of step 1) would be used to color the vertices of K by coloring all the vertices in each of p parts by one of the p colors.

Theorem 19 (Leach and Rodger[2004])

Let $1 \leq r_1 \leq r_2 \leq \ldots \leq r_p$ and G be the multigraph on the p vertices $v_1, \ldots, v_p$ in which v_i is joined to v_j with $r_i r_j$ edges. Let $\Delta = \sum_{i=2}^{p} r_i$. Suppose there exists a $(\Delta + 1)$-edge-coloring of G in which each vertex v_i is incident with $x_{i,j} \leq r_i$ edges of color j. Then there exists a proper $(\Delta + 1)$-edge-coloring of $K(r_1, \ldots, r_p)$ in which for $1 \leq i \leq p$) each vertex in V_i is incident with exactly $\lfloor x_{ij}/r_i \rfloor$ or $\lceil x_{ij}/r_i \rceil$ edges colored j. (So each color class is a matching.)

Complete 5-partite Graphs

Theorem 20 (Dalal and Rodger[2014])

Let $K(r_1, r_2, \ldots, r_5)$ be a complete 5-partite graph of even order. Then K is of Type 2 if and only if def(K) is less than the number of parts of odd size.

Complete 6-partite Graphs

Using previous known results (Theorems 10 (deficiency) , 11 ($r_1 < r_2$), 13 ($r_2 < r_3$) and 14 ($r_3 < r_4$)), the following cases were required to be settled.

1. $K(r, r, r, r, r+2, r+2)$
2. $K(r, r, r, r, r+1, r+3)$
3. $K(r, r, r, r, r, r+4)$
4. $K(r, r, r, r, r, r+2)$
5. $K(r, r, r, r, r+1, r+1)$

$K(r, r, r, r, r, r + 4)$, **when** r **is odd**

- This subcase requires special attention as any total coloring of K in which all the vertices of V_6 are colored the same would leave at least one vertex $v \in V_1 \cup \ldots \cup V_5$ unsaturated.
- However, all the vertices in $V_1 \cup \ldots \cup V_5$ are of the maximum degree, and thus such a total coloring would require at least $\Delta(K) + 2$ colors.
- The approach by Dalal and Rodger [Graphs and Combinatorics (2014)], which requires coloring of all the vertices in same part with the same color, would not yield a $\Delta(K) + 1$ total coloring.

Complete 6-partite Graphs

Let K' be an amalgamation of K (where $V(K') = \{1, 2, \ldots, 7\}$) with amalgamation function $\psi : V(K) \to V(K')$ as defined below:

- $\psi(V_i) = \{i\}$ for $1 \leq i \leq 5$, $\psi(V_6 \setminus \{a\}) = \{6\}$, and $\psi(\{a\}) = 7$, where a is an arbitrary vertex in V_6.
- The associated *number function* $\eta : V(K') \to \mathbb{N}$ thus is defined as: $\eta(\{i\}) = r$ for $1 \leq i \leq 5$, $\eta(\{6\}) = r + 3$ and $\eta(\{7\}) = 1$.

Lemma 1

For odd $r \geq 3$, there exists an equitable 5-edge-coloring of $(r-1) * K_6 + 4 * K[\{1,2,3,4,5\},\{6\}] + K[\{1,2,3,4,5\},\{7\}]$, where $V(K_6) = \mathbb{N}_6$, such that the degree sequence of each color class is $(r, r, r, r, r, r+3, 1)$.

Complete 6-partite Graphs

Let $V(K_6) = \{1, \ldots, 6\}$ and let C_i be the set of edges colored c_i for $1 \leq i \leq \Delta(K) + 1 = 5r + 5$, defined as follows: (note that each color class is majorized by $(\eta(1), \ldots, \eta(6), \eta(7)) = (r, r, r, r, r, r + 3, 1)$).

1. Let $\{W_1, \ldots, W_6\}$ be a decomposition of $2 * K_6$ into 6 cycles of length 5 with vertex i missing in W_i for $1 \leq i \leq 6$ (this exists - refer [BHMS11]).
   - $C_1 = \frac{r-1}{2} * W_1 + K[\{2, 3, 4, 5\}, \{6\}]$
   - $C_2 = \frac{r-1}{2} * W_2 + K[\{1, 3, 4, 5\}, \{6\}] \setminus \{5, 6\} + \{5, 7\}$
   - $C_3 = \frac{r-1}{2} * W_3 + K[\{1, 2, 4, 5\}, \{6\}] \setminus \{4, 6\} + \{4, 7\}$
   - $C_4 = \frac{r-1}{2} * W_4 + K[\{1, 2, 3, 5\}, \{6\}] \setminus \{3, 6\} + \{3, 7\}$
   - $C_5 = \frac{r-1}{2} * W_5 + K[\{1, 2, 3, 4\}, \{6\}] \setminus \{2, 6\} + \{2, 7\}$.
     $C_6 = \frac{r-1}{2} * W_6 + \{1, 7\} + [\{2, 4\}, \{3, 5\}]$.
2. Use Lemma 1 to produce (5)-color classes $M_1, M_2, \ldots, M_5$ in K' such that $\cup_{i=1}^{5} M_i$ induces
   $(r - 1) * K_6 + 4 * K[\{1, 2, 3, 4, 5\}, \{6\}] + K[\{1, 2, 3, 4, 5\}, \{7\}]$.

Complete 6-partite Graphs

2. Take $(r-1)$ copies of each of $M_1, \ldots, M_5$ to form $(5r-5)$ new color classes.
3. Let C_1 and C_2 be a decomposition of K_5 into cycles of length 5 where $C_1 = [\{2,4\},\{3,5\},\{1,4\},\{2,3\},\{1,5\}]$ and $C_2 = [\{3,4\},\{1,3\},\{2,5\},\{4,5\},\{1,2\}]$. Define the 4 color class as follows:
   - $C_{5r+2} = \{5,6\} + \{1,4\} + \{2,3\} + \frac{r-1}{2} * C_1$
   - $C_{5r+3} = \{2,6\} + \{1,5\} + \{3,4\} + \frac{r-1}{2} * C_2$
   - $C_{5r+4} = \{4,6\} + \{1,3\} + \{2,5\} + \frac{r-1}{2} * C_1$
   - $C_{5r+5} = \{3,6\} + \{4,5\} + \{1,2\} + \frac{r-1}{2} * C_2$

Complete 6-paritite Graphs

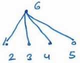
4 Using Bahamanian-Rodger Theorem, disentangle K' to get K with all its edges properly colored using $\Delta(K)+1$ colors $c_1,\ldots,c_{\Delta+1}$.

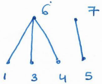
- For $1 \leq i \leq 5$ the color c_i is absent from the vertex $\{i\}$ in K'.
- Therefore, for $1 \leq i \leq 5$, the color c_i is absent from all the vertices in $\psi^{-1}(\{i\}) = V_i \in K$, and we color them with c_i.
- In K', vertex $\{6\}$ is unsaturated by color class C_6 and thus color c_6 is absent from all the vertices in $\psi^{-1}(\{6\}) = V_6 \setminus \{a\}$.
- Color all the vertices in $V_6 \setminus \{a\}$ with c_6.
- In K' the vertex $\{7\}$ is unsaturated by the color class C_{5r+2} and thus color c_{5r+2} is absent from all the vertices in $\psi^{-1}(\{7\}) = \{a\}$.
- Color vertex $a \in V_6$ with c_{5r+2}.

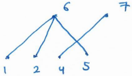
$K(r,r,r,r,r,r+4)$, r is odd

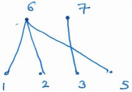
$\underline{\underline{r=3}}$

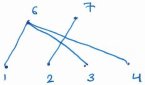
$K(3,3,3,3,3,7)$; $\Delta+1 = 20$

(1) $C_1 = W_1 +$ 

(2) $C_2 = W_2 +$ 

(3) $C_3 = W_3 +$ 

(4) $C_4 = W_4 +$ 

(5) $C_5 = W_5 +$ 

(6) $C_6 = W_5 +$ 

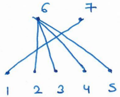
$+$ 

$+$ 

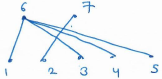
5 matchings of lemma 1

$R_1, \ldots, R_5$ be a 1-factorization of K_6

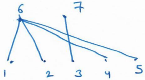
$M_1 = 2 * R_1 +$



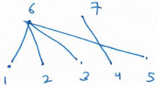
$M_2 = 2 * R_2 +$



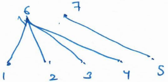
$M_3 = 2 * R_3 +$



$M_4 = 2 * R_4 +$



$M_5 = 2 * R_5 +$



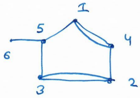
$$\bigcup_{i=1}^{5} M_i = 2 * K_6 + 4 * K[\{1,2,3,4,5\},\{6\}] + K[\{1,2,3,4,5\},\{7\}]$$

We use 10 such matchings to define

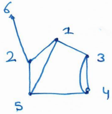
C_7 to C_{16}

Step3

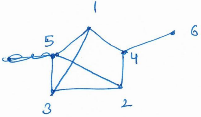
$C_{5r+2} = C_{17} =$



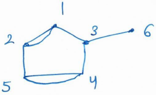
$C_{18} =$



$C_{19} =$



$C_{20} =$



References I

Mehdi Behzad, *Graphs and their chromatic numbers*, Ph.D. thesis, Michigan State University. Department of Mathematics, 1965.

Darryn Bryant, Daniel Horsley, Barbara Maenhaut, and Benjamin R Smith, *Cycle decompositions of complete multigraphs*, Journal of Combinatorial Designs **19** (2011), no. 1, 42–69.

Bor-Liang Chen, Hung-Lin Fu, et al., *Total colorings of graphs of order 2n having maximum degree 2n-2.*, Graphs Comb. **8** (1992), no. 2, 119–123.

KH Chew and HP Yap, *Total chromatic number of complete r-partite graphs*, Journal of graph theory **16** (1992), no. 6, 629–634.

Aseem Dalal, B. S. Panda, and C. A. Rodger, *Total-colorings of complete multipartite graphs using amalgamations*, Discrete Mathematics **339** (2016), no. 5, 1587–1592.

References II

Aseem Dalal and C. A. Rodger, *The total chromatic number of complete multipartite graphs of low deficiency*, (Under Review).

———, *The total chromatic number of complete multipartite graphs with low deficiency*, Graphs and Combinatorics **31** (2015), 2159–2173.

L Dong and H. P. Yap, *The total chromatic number of unbalanced complete r-partite graphs of even order*, Bull. Inst. Comb. Appl **28** (2000), 107–117.

Anthony J. W. Hilton, *A total-chromatic number analogue of plantholt's theorem*, Discrete mathematics **79** (1990), no. 2, 169–175.

DG Hoffman and C. A. Rodger, *The total chromatic number of complete multipartite graphs*, Congressus Numerantium (1996), 205–220.

References III

C. D. Leach and C. A. Rodger, *Hamilton decompositions of complete graphs with a 3-factor leave*, Discrete mathematics **279** (2004), no. 1-3, 337–344.

HP Yap, *Total colourings of graphs*, Bulletin of the London Mathematical Society **21** (1989), no. 2, 159–163.

Thank You