Geometric Covering Number: Covering Points with Curves

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(Joint work with Arijit Bishnu and Mathew Francis)

Indian Statistical Institute

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  1. A point configuration (e.g. grid)
  2. A type of curves (e.g. lines)

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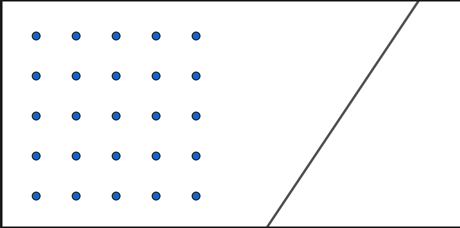

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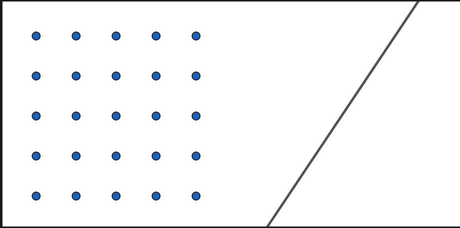

Problem:
Find the minimum number of curves needed to cover the point set.

Covering $n \times n$ grid by lines

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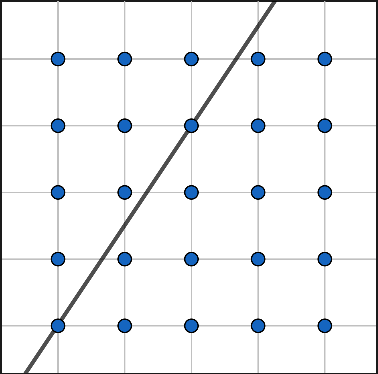

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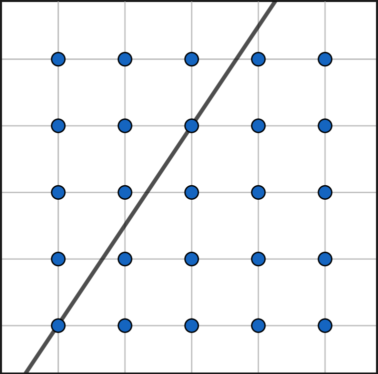

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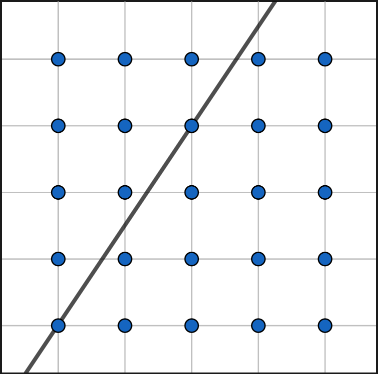

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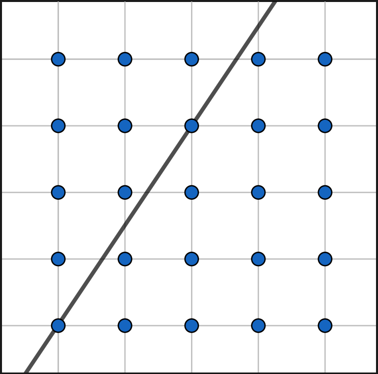

- Any line contains at most n grid points.
- Need at least $n^2/n = n$ lines.
- Also generalizes to higher dimensions (i.e. for covering $n \times \cdots \times n$ grid by lines).

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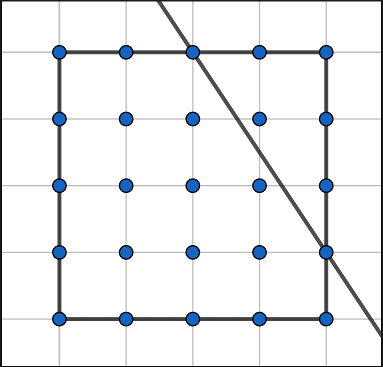

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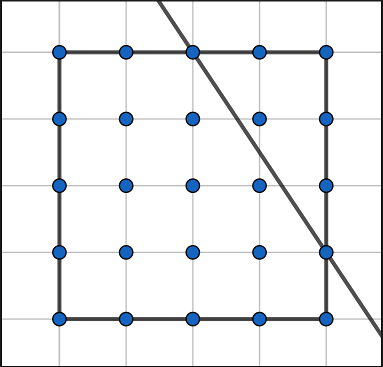

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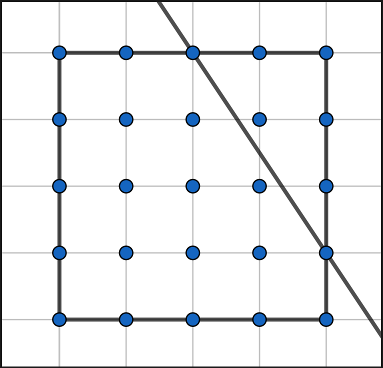

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- Need at least $(4n-4)/2 = 2n-2$ skew lines.

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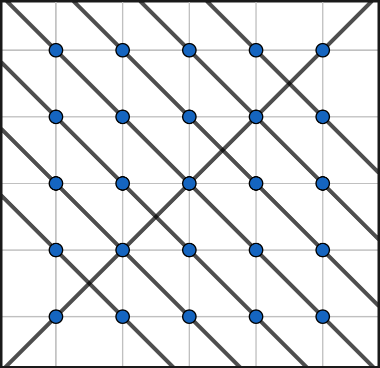

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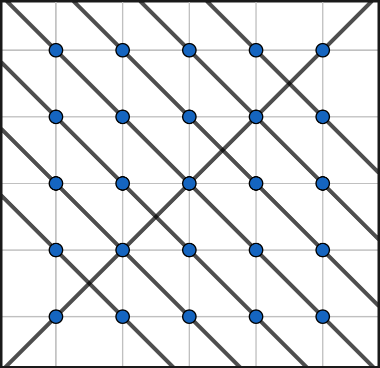

- Number of lines $= 2(n-2) + 2 = 2n - 2$.

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- We show that the answer is NO.

Theorem

There exists a finite set P of n^2 points in $\mathbb{R}^2$ which can be covered with n lines but no subset of P of size $\Omega(n^2)$ can be contained in a projective transformation of a rectangular grid of size $o(n^3)$.

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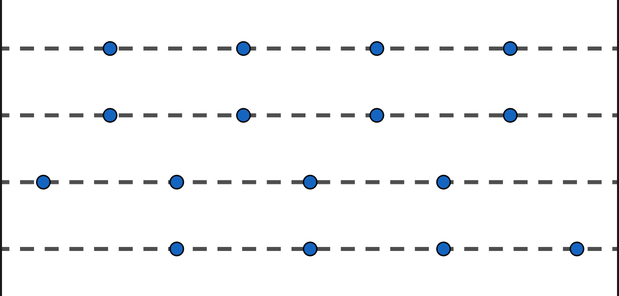

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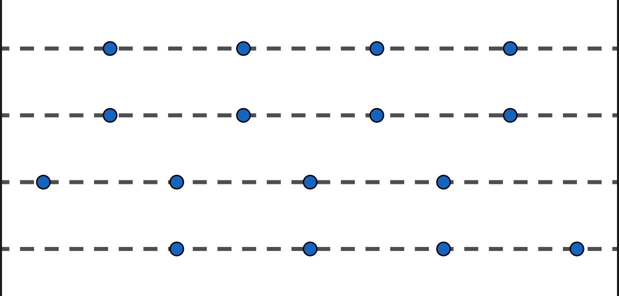

- n parallel lines, each containing n points.

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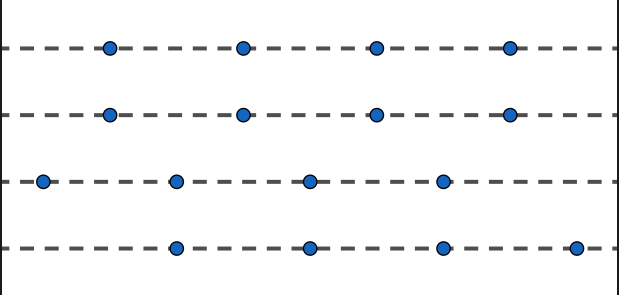

- n parallel lines, each containing n points.
- No three points from three different lines are collinear.

Covering by orthoconvex curves

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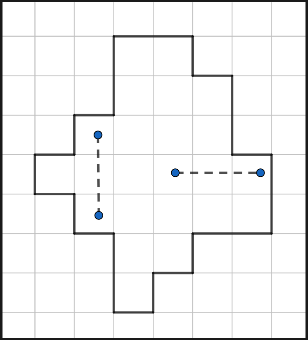
A rectilinear curve is called **orthoconvex** if for any two points (having the same x/y coordinate) lying inside the curve, the vertical/horizontal line segment joining the two points also lies inside the curve.



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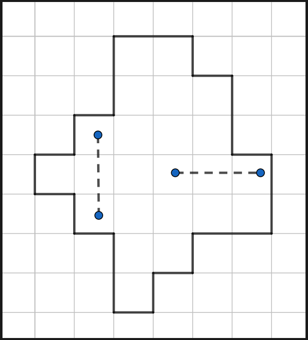

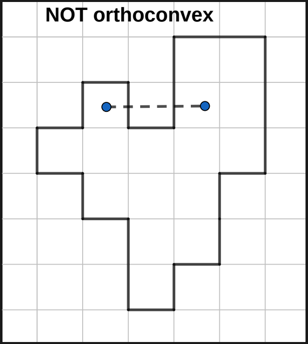

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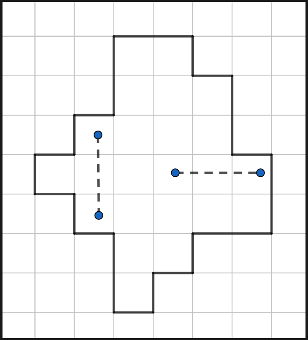

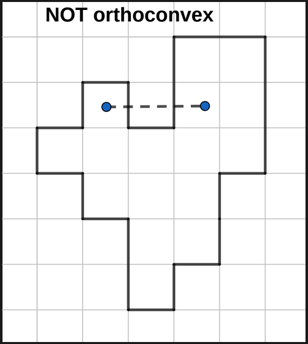

Question: What is the minimum number of orthoconvex curves required to cover an $n \times n$ grid?

Covering $5 \times 5, 6 \times 6$ and 7×7 grid

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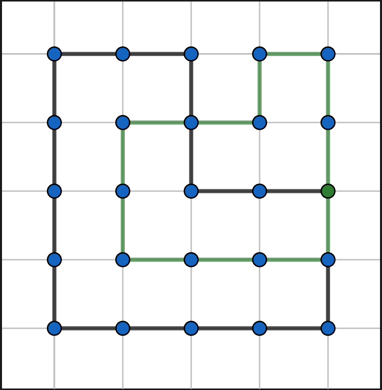

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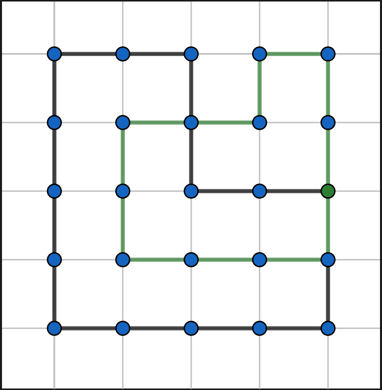

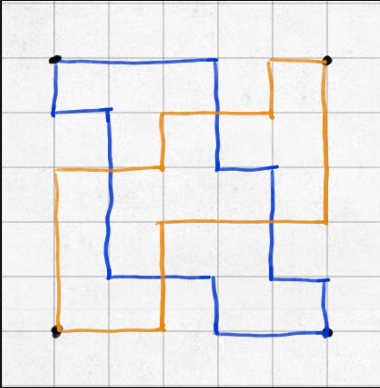

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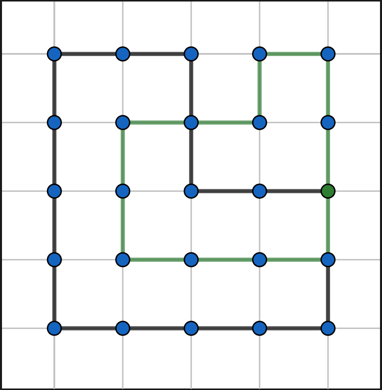

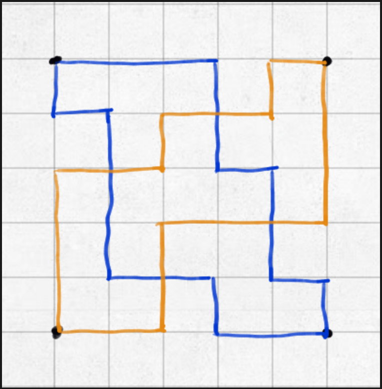

- Both 5×5 and 6×6 grid can be covered by 2 orthoconvex curves.

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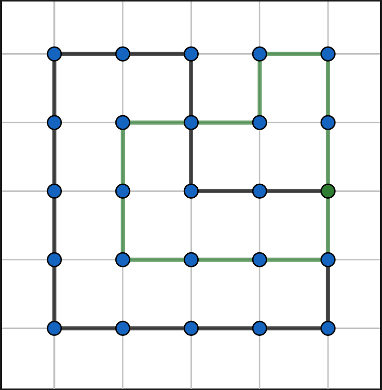

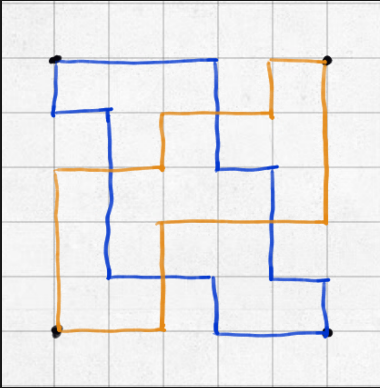

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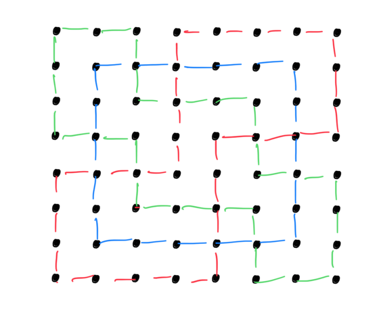
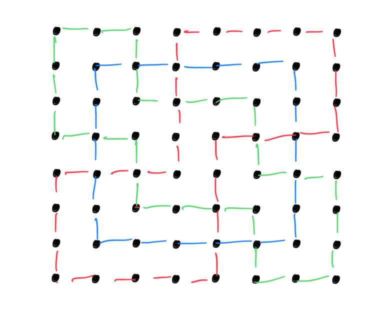
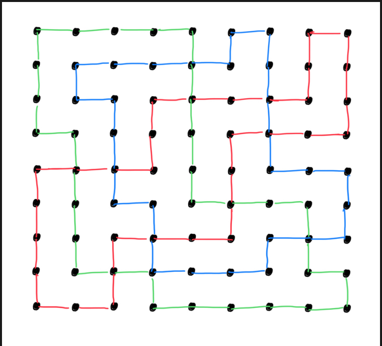


Covering $8 \times 8, 9 \times 9$ and 10×10 grid





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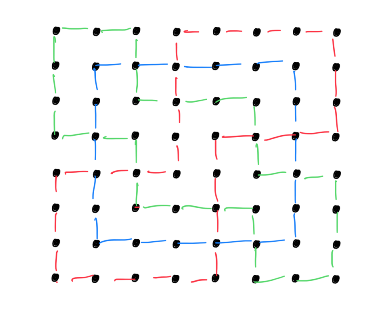

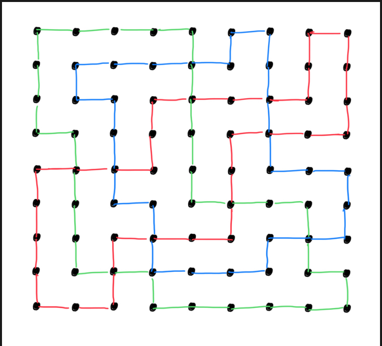

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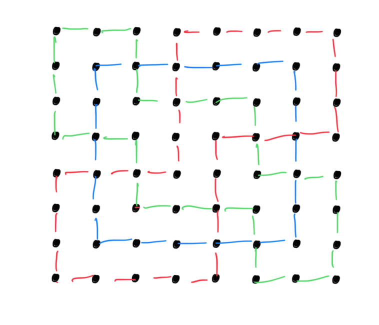

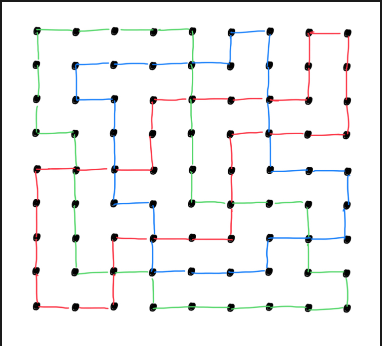

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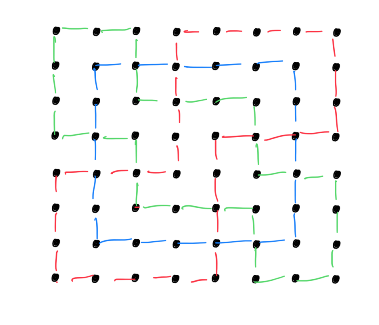

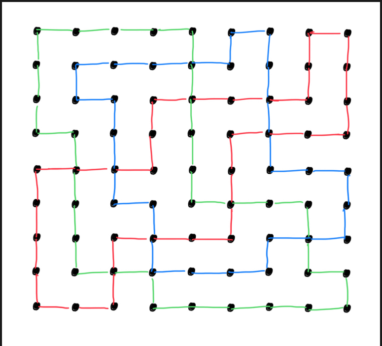

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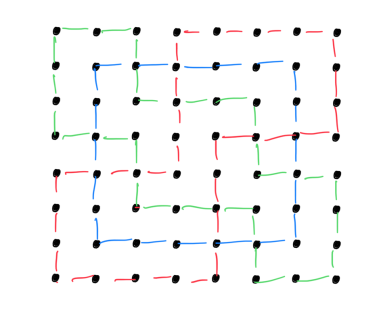

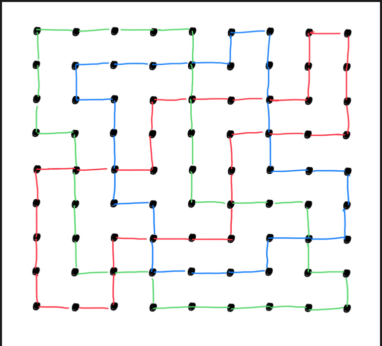

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- 10×10 grid can be covered by 4 orthoconvex curves.
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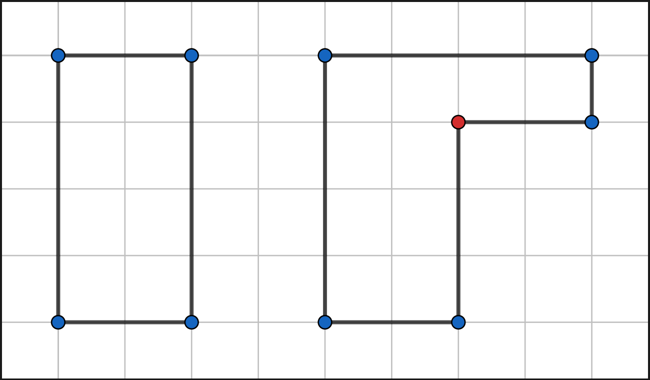

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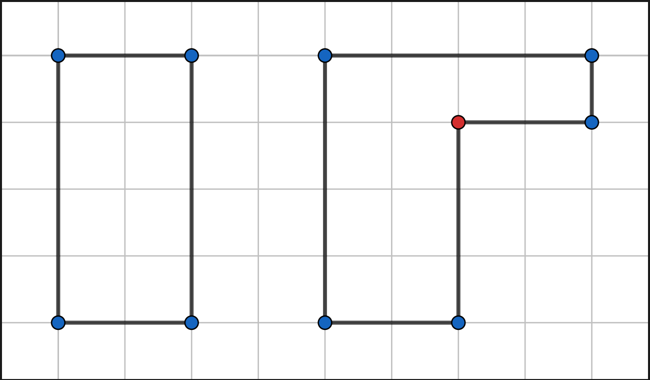

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- Then every curve in C can cover at most two points on this grid line.

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- But then some points on this grid line are not covered by any curve in C, which is a contradiction. □

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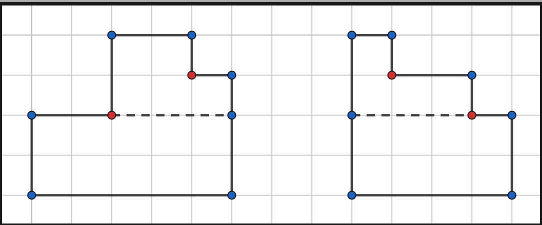

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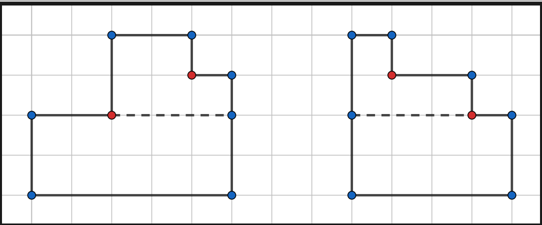

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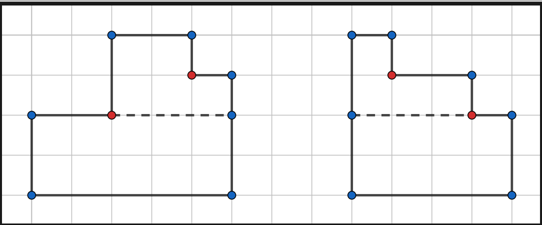

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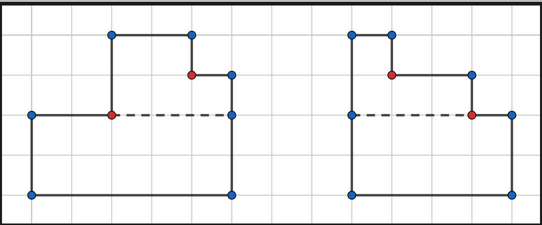

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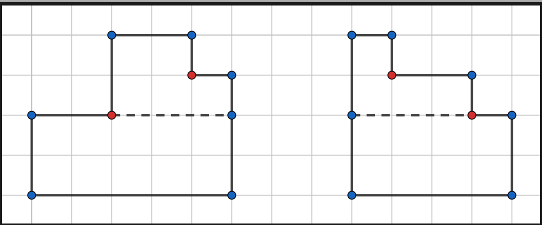

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- Is this bound tight? (we think NO)
- What happens if we have 3 or more inner corners?

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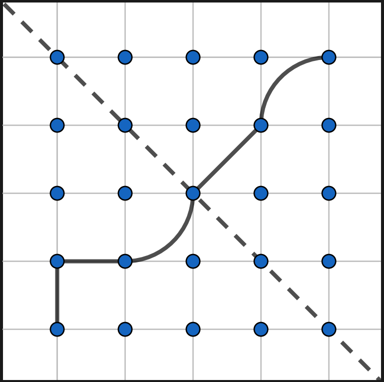
For $n \times n$ grid, any monotonic (or, weakly increasing) curve can intersect the diagonal in at most 1 point.



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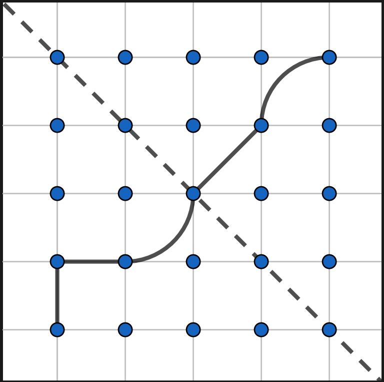

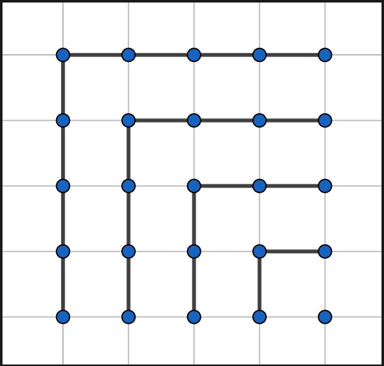

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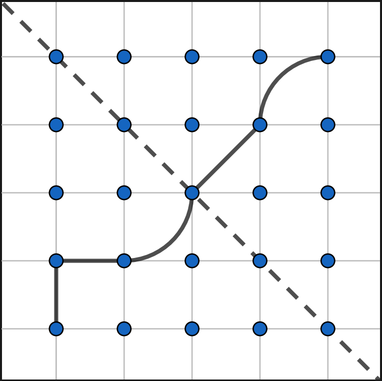

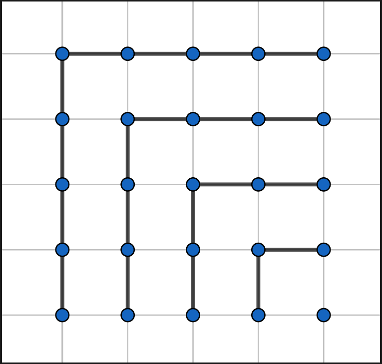

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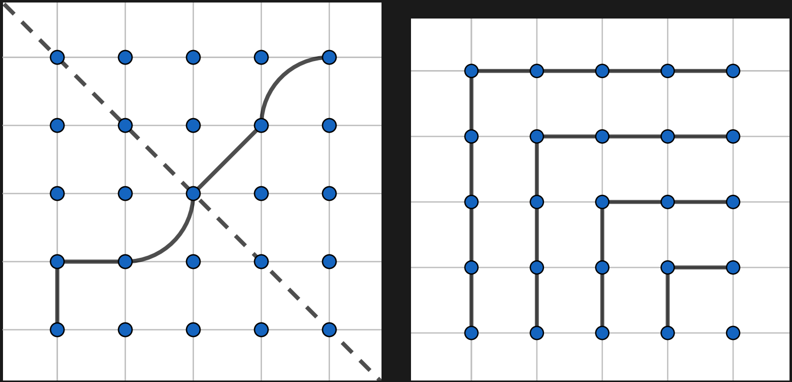

- So minimum number of curves to cover the grid is n.
- How to generalize this to higher dimension and other point configurations?

Definition

Let $f : [0,1] \to \mathbb{R}^d$ be a curve and suppose $f(t) = (f_1(t), \ldots, f_d(t))$ for $t \in [0,1]$. Then f is called **monotonic** if it satisfies the following property:
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- Therefore the problem boils down to covering $\mathcal{P}$ with minium number of chains.

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$$w(\mathcal{P}) = \max_m A_m = A_{\lfloor (k_1 + \cdots + k_d + d)/2 \rfloor},$$

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- A special case: For $P = \{0,1\}^d$ (Hypercube)

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- *Convex curves*: Minimum number of convex curves required to cover an $m \times n$ grid is $\min\{\lceil m/2 \rceil, \lceil n/2 \rceil\}$.

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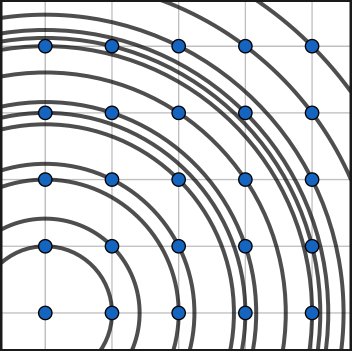

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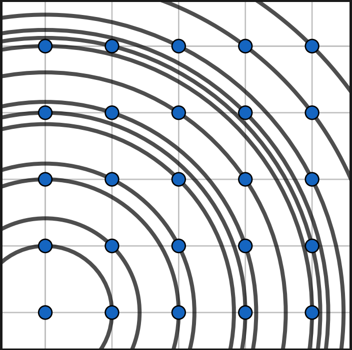

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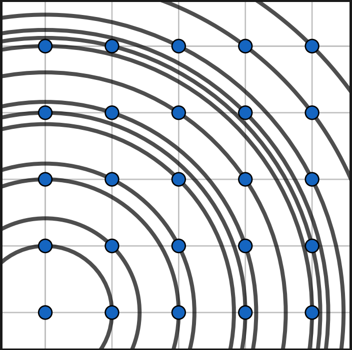

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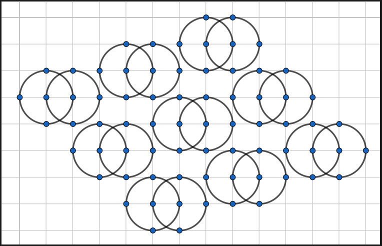

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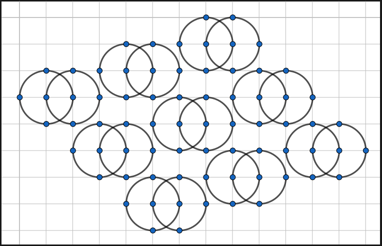

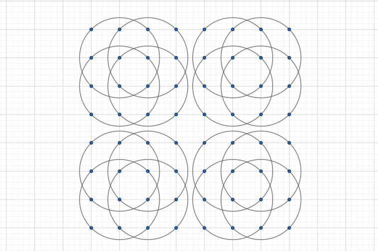

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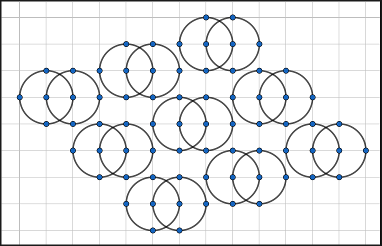

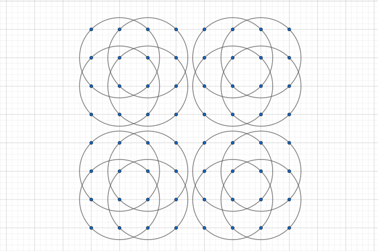

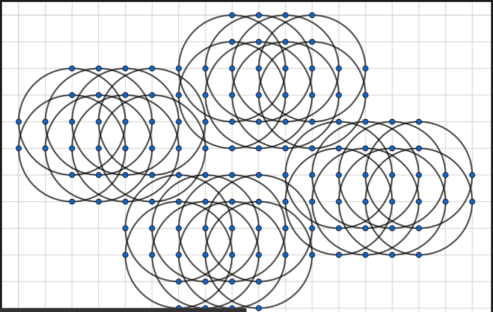

Thank you