# Growth Rate of the Number of Empty Triangles in the Plane

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for any set of points $P$, with $|P| = n$, in general position.

- On the other hand, a set of $n$ points chosen uniformly and independently at random from a convex set of area 1 contains $2n^2 + o(n^2)$ empty triangles on expectation.

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- $K_4 \setminus \{e\}$ : *kite graph*, that is, the complete graph $K_4$ with one of its diagonals removed,
- $\Delta(x, P) = |N_{\triangle}(P) - N_{\triangle}(P \setminus \{x\})|$.

# Example:

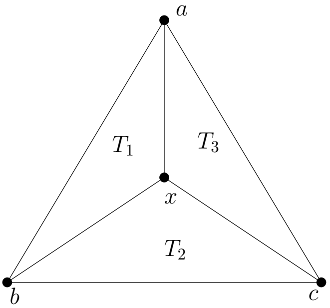


Figure: A set of points $P = \{x, a, b, c\}$

# Theorem 1

**Theorem**

*For any set $P$, with $|P| = n$,*

$$\Delta(x, P) \leq V_P(x) + H(V_P(x), K_3, K_4 \setminus \{e\}),$$

*where $H(V_P(x), K_3, K_4 \setminus \{e\})$ is the maximum number of triangles in a $K_4 \setminus \{e\}$-free graph on $V_P(x)$ vertices.*

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*Moreover, there exists a set* $P$*, with* $|P| = n$*, and a point* $x \in P$ *such that* $\Delta(x, P) \geq CV_P(x)^{\frac{3}{2}}$*, for some constant* $C > 0$.

# Proof of the Upper Bound

**Observation 1**

$\Delta(x, P) \leq V_P(x) + I_P(x)$, where $I_P(x)$ is the number of triangles in $P$ that contain only the point $x$ in the interior.

To prove Theorem 1 we have to show

$$I_P(x) \leq H(V_P(x), K_3, K_4 \setminus \{e\})$$

# Transformation of Geometric problem into Graph problem

Given a set $P$, with $|P| = n$, and a point $x \in P$, define the graph $G_P(x)$ as follows:

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We call the graph $G_P(x)$ the *empty triangle graph incident at $x$*.

# Transformation of Geometric problem into Graph problem:

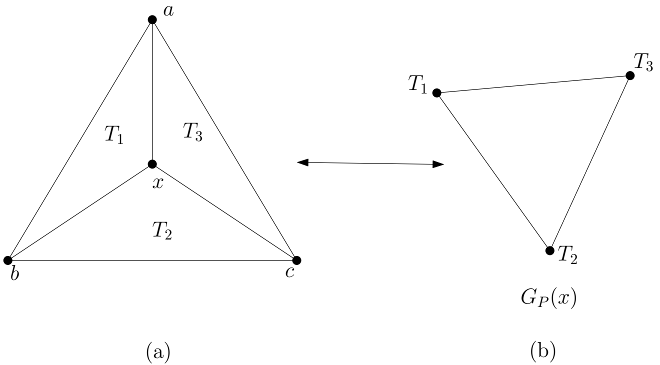


Figure: A set of points $P = \{x, a, b, c\}$ and the empty triangle graph incident at $x$.

# Proof Approach

## Observation 2

Suppose $P$ be a set of points in the plane, with $|P| = n$, in general position and $x \in P$. Then

$$I_P(x) = N_{K_3}(G_P(x)),$$

where $N_{K_3}(G_P(x))$ is the number of triangles in the graph $G_P(x)$.

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**Observation 3**

The graph $G_P(x)$ does not contain $K_4 \setminus \{e\}$ as a subgraph, that is, $G_P(x)$ is kite-free.

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# Proof of the Upper Bound

From previous observations, it follows that

$$\Delta(x, P) \leq V_P(x) + I_P(x) = V_P(x) + N_{K_3}(G_P(x)).$$

and

$$N_{K_3}(G_P(x)) \leq H(V_P(x), K_3, K_4 \setminus \{e\})$$

and thus the upper bound in Theorem 1, is proved i.e

$$\Delta(x, P) \leq V_P(x) + H(V_P(x), K_3, K_4 \setminus \{e\})$$

# Lower Bound Construction

- Consider the set of points $P$, with $|P| = n = 3L + 1$.
- $P$ consists of three point sets $A$, $B$, and $C$, with $|A| = |B| = |C| = L$, arranged along 3 disjoint convex chains and a point $x$ at the middle.

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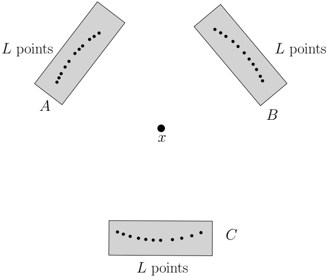


Figure: Example showing the lower bound in Theorem 1.

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## Observation 6

$U_P(x) = 3\binom{L}{3} + 6L\binom{L}{2} \sim 3.5L^3$

where $U_P(x) =$ the number of empty triangles in $P$ such that $x$ is not a vertex of the empty triangles.

# Lower Bound Construction

## Result

- $N_\Delta(P) = V_P(x) + U_P(x)$
- $N_\Delta(P) \sim 3.5L^3 + \Theta(L^2)$
- $\Delta(x, P) = |N_\Delta(P) - N_\Delta(P \setminus \{x\})| = \Theta(L^3) = \Theta(V_P(x)^{\frac{3}{2}})$

# Some Geometric Properties of the Graph $G_P(x)$

- Fix $r, s \geq 1$. Then there exists a set of points $P$ and $x \in P$ such that the graph $G_P(x)$ contains the complete bipartite graph $K_{r,s}$.

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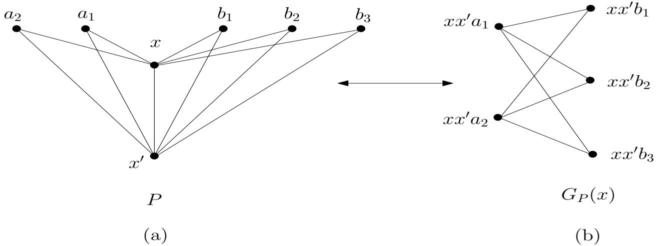


Figure: Bipartite subgraph

# Polynomial improvement of the upper bound of $\Delta(x, P)$

## Kővári-Sós-Turán theorem

Let $K_{r,s}$ be a complete bipartite graph with $r \leq s$ then,

$$ex(n, K_{r,s}) = O(n^{2-\frac{1}{r}})$$

where $ex(n, H)$ is the maximum number of edges in a graph with $n$ vertices which does not contain a copy of graph $H$.

- As $G_P(x)$ can contain complete bipartite graph as a subgraph, we cannot apply Kővári-Sós-Turán theorem to improve the upper bound on $\Delta(x, P)$.

# Polynomial improvement of the upper bound of $\Delta(x, P)$

**Szemeredi's theorem**

If $H$ is an abelian group with $n$ elements and $A$ is a subset of $H$ with no length three arithmetic progressions then we can construct a graph $G = (V, E)$ that has $3n$ vertices and $|A|.n$ pairwise edge disjoint triangles and no other triangles.

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- Combining this two results it is possible to construct a graph with $3n$ vertices and $n^2/e^{O(\sqrt{\log n})}$ edge disjoint triangles and no other triangles.

# Polynomial improvement of the upper bound of $\Delta(x, P)$

**Behrend's construction:**

Suppose $n$ is an odd prime and $A \subseteq \mathbb{Z}/n\mathbb{Z}$ is a set with no 3-term arithmetic progression. The Behrend's graph $G(n, A)$ is a tripartite graph with vertices on each side of the tripartition numbered $\{0, 1, \ldots, n-1\}$ and triangles of the form $(z, z+a, z+2a)$ modulo $n$, for $z \in \{0, 1, \ldots, n-1\}$ and $a \in A$.

- It is easy to check that the graph $G(n, A)$ has $3n$ vertices $3|A|n$ edges and each edge belongs to a unique triangle.

# Polynomial improvement of the upper bound of $\Delta(x, P)$

## Behrend's construction: Example

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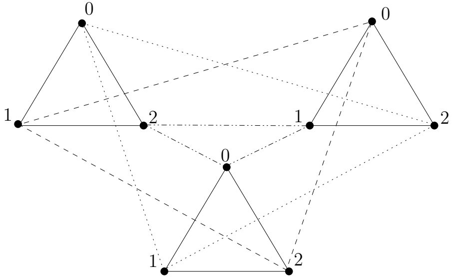


Figure: The Paley graph with 9 vertices, 18 edges, and 6 triangles.

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- From Behrend's graph one can get a lower bound of $\Omega(n^2/e^{O(\sqrt{\log n})})$ for the Ruzsa-Szemerédi problem(that asks maximum number of edges in a graph in which every edge belongs to a unique triangle) and hence, for $H(n, K_3, K_4 \setminus \{e\})$.

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- But the previous Paley graph cannot be geometrically realized, that is, it is not possible to find a set of points $P$ and $x \in P$ such that $G_P(x)$ is isomorphic to the previous Paley graph.

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- This, in particular, means that there are 6 triangles in $P$ which only contains the point $x$ in the interior. Denote these triangles by $\mathcal{T} = \{T_1, T_2, \ldots, T_6\}$.

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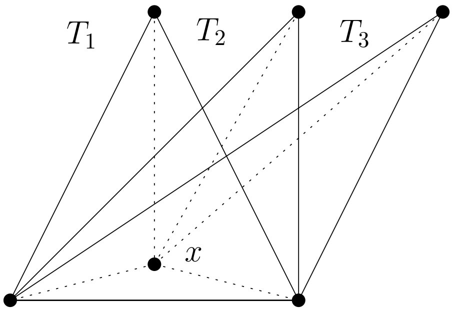
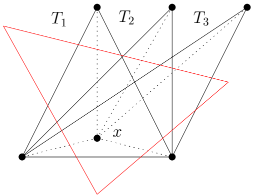
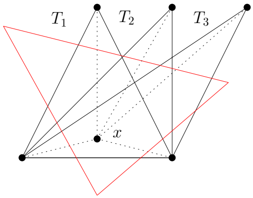





(a) $V_P(x) = 10, I_P(x) = 4$

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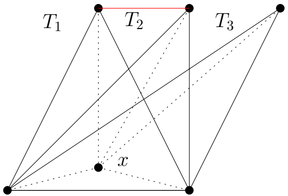


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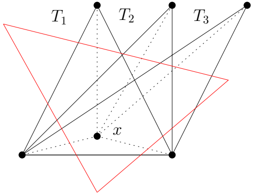


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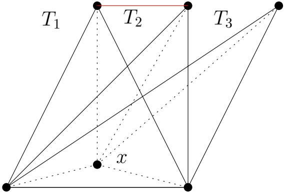


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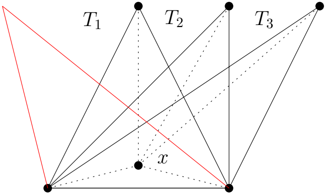


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# Polynomial improvement of the upper bound of $\Delta(x, P)$

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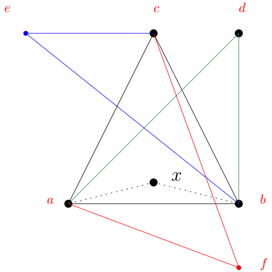

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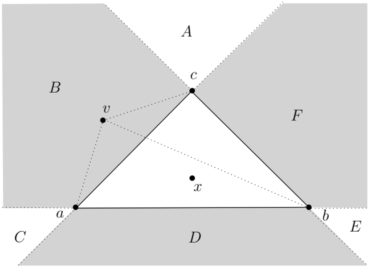


Figure

# Polynomial improvement of the upper bound of $\Delta(x, P)$

- To get $V_P(x) = 9$ and $I_P(x) = 6$ each edge of of all triangles in $T$ must be shared by more than one triangles of $T$
- By Claim 1 and Claim 2, Paley graph with 9 vertices, 18 edges, and 6 triangles is not geometrically realizable.

# Conclusion

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- Our results show that $\Delta(x, P)$ can range from $O(V_P(x)^{\frac{3}{2}})$ and $o(V_P(x)^2)$.
- Understanding additional properties of the graph $G_P(x)$ is an interesting future direction, which can be useful in improving the bounds on $\Delta(x, P)$.

# Thank you