Location-domination type problems under the Mycielski construction

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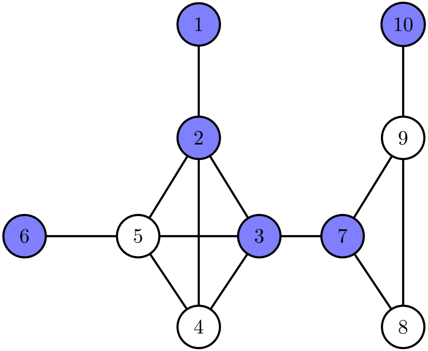
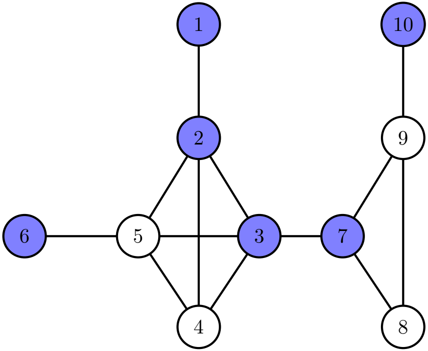
Location-domination in graphs

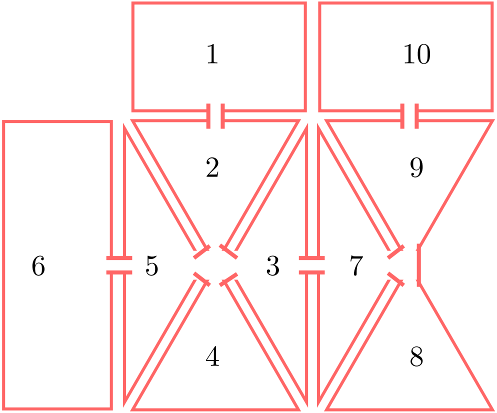


Location-domination in graphs



- $4 \longleftrightarrow \{2,3\}$
- $5 \longleftrightarrow \{2,3,6\}$
- $8 \longleftrightarrow \{7\}$
- $9 \longleftrightarrow \{7,10\}$

Locating-dominating (LD) set – A practical example



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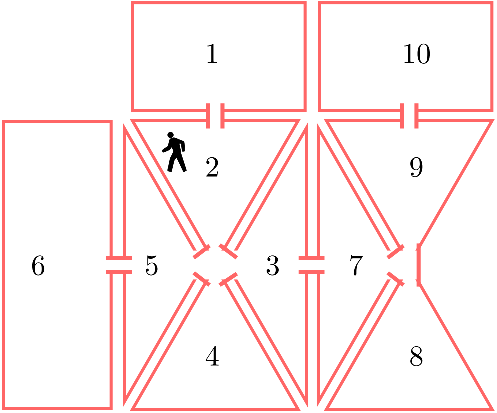

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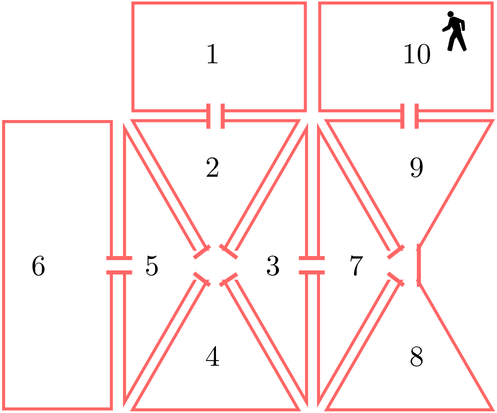

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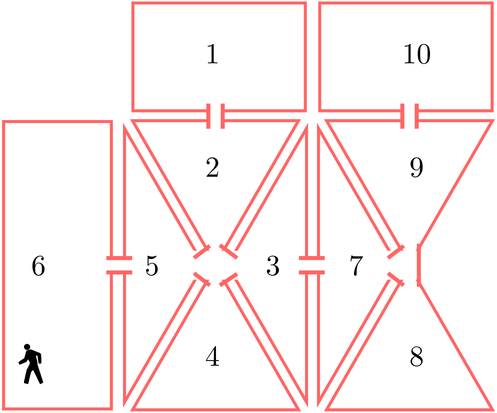

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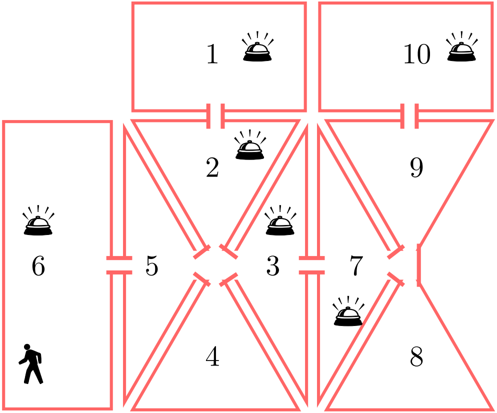

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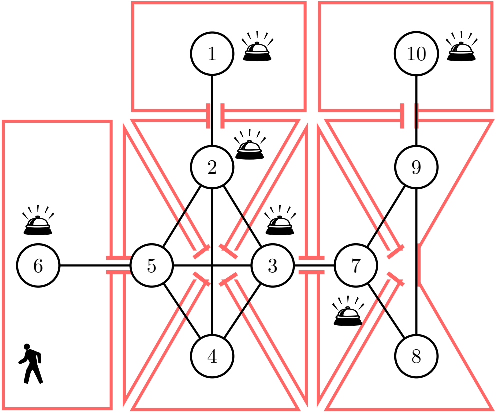

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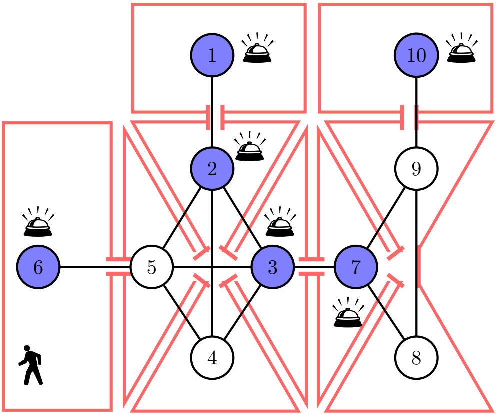

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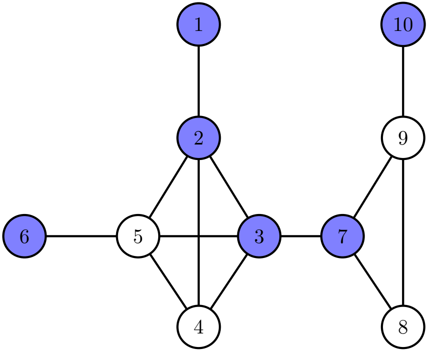

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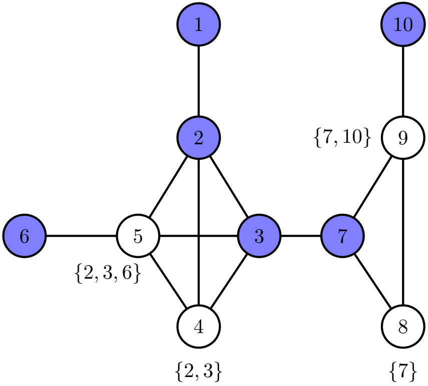

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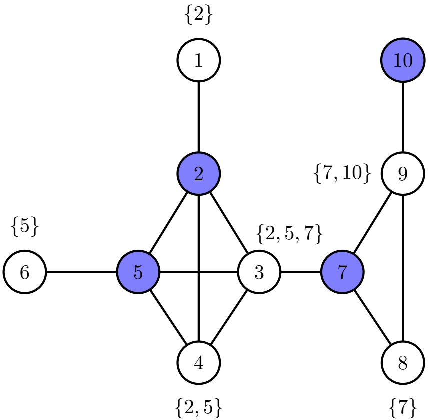

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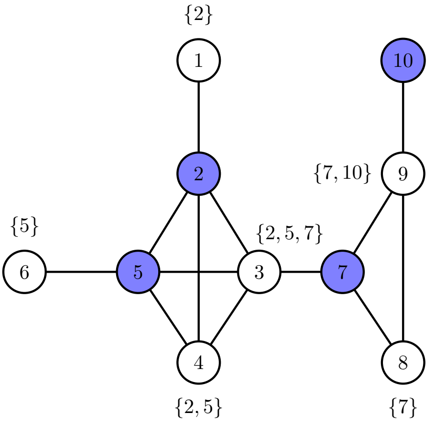
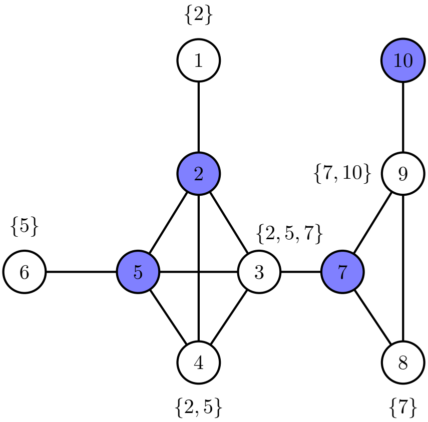


$S = \{\text{blue vertices}\}$

- $1 \longleftrightarrow \{2\} = N(1) \cap S$
- $3 \longleftrightarrow \{2, 5, 7\} = N(3) \cap S$
- $4 \longleftrightarrow \{2, 5\} = N(4) \cap S$
- $6 \longleftrightarrow \{5\} = N(6) \cap S$
- $8 \longleftrightarrow \{7\} = N(8) \cap S$
- $9 \longleftrightarrow \{7, 10\} = N(9) \cap S$

Locating-dominating (LD) set – A practical example



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S has two properties:

1. Domination: “Well spread-out”
2. Separation (*location*):

$N(u) \cap S \neq N(v) \cap S, \forall u, v \notin S$

Definitions (Neighbourhoods)

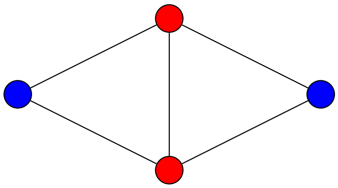
Let $G = (V, E)$ be a graph...

Neighbourhoods

open: $N(v) = \{u \in V : uv \in E\}$. || *closed*: $N[v] = N(v) \cup \{v\}$.

Twins

$$u, v \in V \text{ are called } \begin{Bmatrix} \text{open twins} \\ \text{closed twins} \end{Bmatrix} \iff \begin{Bmatrix} N(u) = N(v) \\ N[u] = N[v] \end{Bmatrix}.$$



Diamond

Definitions (Domination)

Let $G = (V, E)$ be a graph...

Dominating set [property: *closed domination*]

$S \subset V$ such that $N[v] \cap S \neq \emptyset$ for all $v \in V$.

A dominating set always exists.

Open/total dominating set [property: *open domination*]

$S \subset V$ such that $N(v) \cap S \neq \emptyset$ for all $v \in V$.

An open dominating set exists $\iff$ G is isolate-free.

Definitions (Separation)

Let $G = (V, E)$ be a graph...

Locating set [property: *location*]

$S \subset V$ such that $N(u) \cap S \neq N(v) \cap S$ for all $u, v \in V \setminus S$.

Open separating set [property: *open separation*]

$S \subset V$ such that $N(u) \cap S \neq N(v) \cap S$ for all distinct $u, v \in V$.

Open separating set exists $\iff$ G is open-twin free.

Closed separating set [property: *closed separation*]

$S \subset V$ such that $N[u] \cap S \neq N[v] \cap S$ for all distinct $u, v \in V$.

Closed separating set exists $\iff$ G is closed-twin free.

Definitions: Let $G = (V, E)$ be a graph...

Locating-dominating (LD) set [1] X=LD

$S \subset V$ is a *locating-dominating set* if S is a

- dominating set
- locating set: $N(u) \cap S \neq N(v) \cap S$ for all distinct $u, v \in V \setminus S$

LD code always exists.

[1] Slater, 1988

Definitions: Let $G = (V, E)$ be a graph...

Locating total-dominating (LTD) set [2] X=LTD

$S \subset V$ is a *locating total-dominating set* if S is a

- total-dominating set
- locating set: $N(u) \cap S \neq N(v) \cap S$ for all distinct $u, v \in V \setminus S$

LTD code exists $\iff$ G is isolate-free.

[2] Haynes, Henning & Howard, 2006.

Definitions: Let $G = (V, E)$ be a graph...

Open-locating dominating (OLD) set [3] X=OLD

$S \subset V$ is an *open locating-dominating set* if S is a

- total-dominating set
- open-separating set: $N(u) \cap S \neq N(v) \cap S$ for all distinct $u, v \in$

OLD code exists $\iff$ G is open-twin free and isolate-free.

[3] Seo & Slater, 2010.

X-sets studied in literature.

Code name	**X**	**dom(X)**	**sep(X)**	$\gamma^{\mathrm{X}}(\mathbf{G})$
Identifying	ID	CD	CS	$\gamma^{\mathrm{ID}}(G)$
Open-Separating Dominating	OSD	CD	OS	$\gamma^{\mathrm{OSD}}(G)$
Locating- Dominating	LD	CD	L	$\gamma^{\mathrm{LD}}(G)$
Differentiating Total-Dominating	DTD	OD	CS	$\gamma^{\mathrm{DTD}}(G)$
Open-Locating Dominating	OLD	OD	OS	$\gamma^{\mathrm{OLD}}(G)$
Locating Total-Dominating	LTD	OD	L	$\gamma^{\mathrm{LTD}}(G)$

CD : closed domination OD : open domination
CS : closed separation OS : open separation L : location

X-number

$\gamma^{\mathrm{X}}(G) = \min\{|S| : S \text{ is an X-set of } G\}$.

Mycielski Construction

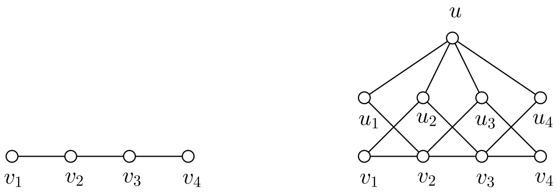


Figure: The path P_4 and the resulting graph $M(P_4)$

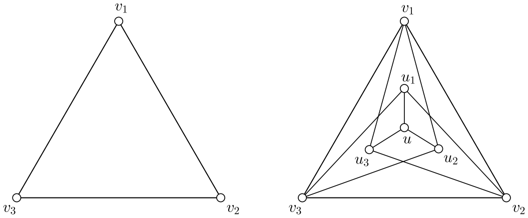


Figure: The cycle C_3 and the resulting graph $M(C_3)$

Lower bounds on $\gamma^X(G)$

Theorem

Let $X \in \{LD, LTD, OLD\}$. For a graph G that is either a path P_n or a cycle C_n admitting an X-set, we have

$$\gamma^X(M(G)) \geq \gamma^X(G) + 1.$$

Theorem (Bertrand, Charon, Hudry & Lobstein, 2004)

If G equals P_n or C_n for $n \geq 3$, we have as lower bound:

$$\gamma^{LD}(G) = \left\lceil \frac{2n}{5} \right\rceil.$$

Theorem (Henning & Rad, 2014)

If G equals P_n or C_n for $n \geq 3$, we have as lower bound:

$$\gamma^{LTD}(G) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil.$$

Theorem (Seo & Slater, 2010)

For P_n with $n = 6k + r$ for $k \geq 1$ and $r \in \{0, \ldots, 5\}$, then we have:

$$\gamma^{OLD}(P_n) = \begin{cases} 4k + r & \text{if} \quad r \in \{0, \ldots, 4\}, \\ 4k + 4 & \text{if} \quad r = 5; \end{cases}$$

Theorem (Bianchi, C., Lucarini, Wagler, 2024+)

For C_n *with* $n = 6q + r$ *such that* $n \geq 3$ *and* $n \neq 4$, *we have*

$$\gamma^{OLD}(C_n) = \begin{cases} 4q + r, & \text{if } r = 0, 1, 2, 4, \\ 4q + r - 1, & \text{if } r = 3, 5. \end{cases}$$

*Note: Above theorem is a correction of the result from the literature.

Corollary

For C_n *with* $n = 6q + r$ *such that* $n \geq 3$ *and* $n \neq 4$, *we have*

$$\gamma^{OLD}(M(C_n)) \geq \begin{cases} 4q + r + 1, & \text{if } r = 0, 1, 2, 4, \\ 4q + r, & \text{if } r = 3, 5. \end{cases}$$

Upper bounds for $\gamma^{OLD}(G)$

Theorem

Let G be a graph without isolated vertices and open twins. Then

$$\gamma^{OLD}(M(G)) \leq \gamma^{OLD}(G) + 2.$$

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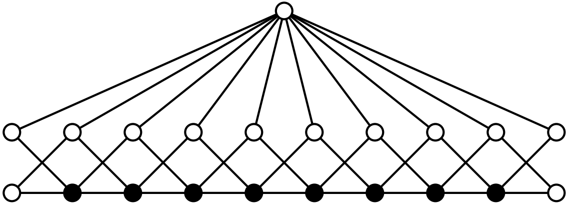

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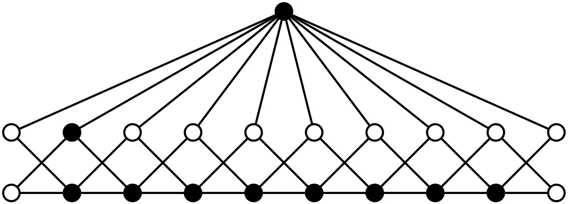

Figure: $\gamma^{OLD}(M(P_{10})) \leq 10 = \gamma^{OLD}(P_{10}) + 2.$

Upper bounds for $\gamma^{LD}(M(G))$ and $\gamma^{LTD}(G)$

Theorem

Let $X \in \{LD, LTD\}$. For a graph G admitting an X-set, we have

$$\gamma^X(M(G)) \leq 2\gamma^X(G).$$

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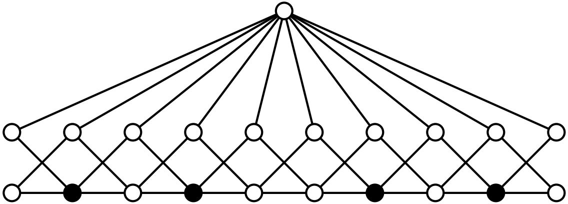

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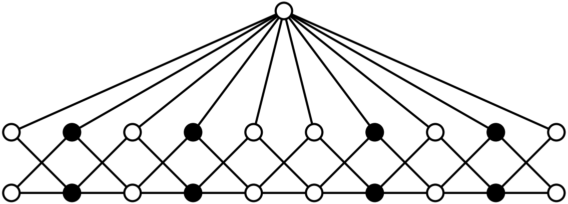

Figure: $\gamma^{LD}(M(P_{10})) \leq 8 = 2 \cdot \gamma^{LD}(P_{10})$.

Upper bounds for $\gamma^{LD}(M(G))$ and $\gamma^{LTD}(G)$

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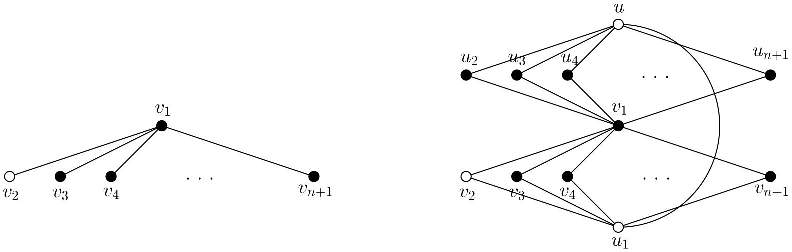

Figure: $K_{1,n}$ and $M(K_{1,n})$. Black vertices depict LD- & LTD-codes.

Upper bounds for $\gamma^{LD}(P_n)$ and $\gamma^{LD}(C_n)$

Theorem

For P_n *with* $n = 3k + r$, *where* $k \geq 2$ *and* $r \in \{0, 1, 2\}$, *we have:*

$$\gamma^{LD}(M(P_n)) \leq \begin{cases} 2k+1 & \text{if} \quad r = 0 \\ 2k+2 & \text{if} \quad r \in \{1,2\} \end{cases}$$

and, for C_n *with* $n = 6k + r$, *where* $k \geq 2$ *and* $r \in \{0, \ldots, 5\}$, *we have:*

$$\gamma^{LD}(M(C_n)) \leq \begin{cases} 4q + r + 1, & \text{if } r = 0, 1, 2, 4, \\ 4q + r, & \text{if } r = 3, 5. \end{cases}$$

Upper bounds for $\gamma^{LD}(P_n)$ and $\gamma^{LD}(C_n)$

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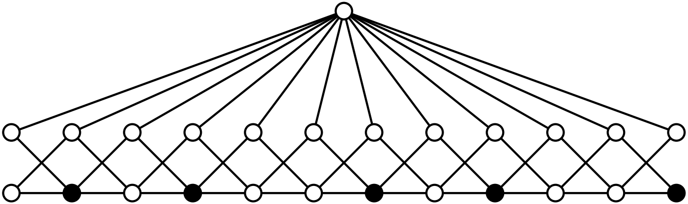

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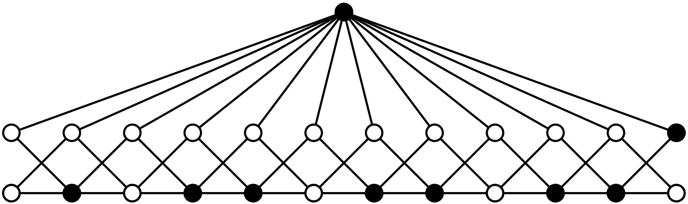


Figure: $\gamma^{LD}(M(P_{12})) \leq 9$.

Upper bounds for $\gamma^{LTD}(P_n)$ and $\gamma^{LTD}(C_n)$

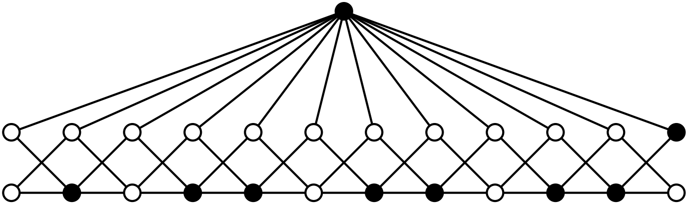
Theorem

For P_n with $n = 6k + r$ for $k \geq 1$, $r \in \{0, \ldots, 5\}$ and C_n with $n \geq 3$, we have:

$$\gamma^{LTD}(M(P_n)) \leq \begin{cases} 4k+2 & \text{if } r = 0 \\ 4k+r+1 & \text{if } r \in \{1,2,3\} \\ 4k+r & \text{if } r \in \{4,5\} \end{cases}$$

and

$$\gamma^{LTD}(M(C_n)) \leq \begin{cases} 4q+r+2, & \text{if } r = 0,1,2,4, \\ 4q+r+1, & \text{if } r = 3,5. \end{cases}.$$



Upper bounds for $\gamma^{LTD}(P_n)$ and $\gamma^{LTD}(C_n)$

Theorem

For P_n *with* $n = 6k + r$ *for* $k \geq 1$, $r \in \{0, \ldots, 5\}$ *and* C_n *with* $n \geq 3$, *we have:*

$$\gamma^{LTD}(M(P_n)) \leq \begin{cases} 4k+2 & \textit{if} \quad r = 0 \\ 4k+r+1 & \textit{if} \quad r \in \{1,2,3\} \\ 4k+r & \textit{if} \quad r \in \{4,5\} \end{cases}$$

and

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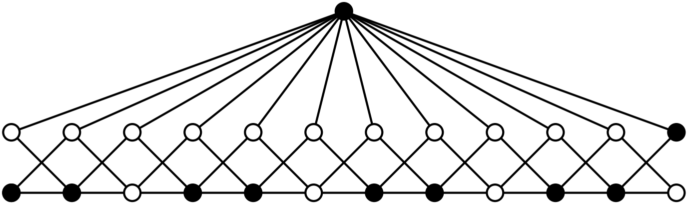

Figure: $\gamma^{LTD}(M(P_{12})) \leq 10$.

Possible future research ideas...

- Increase the lower bound!
- Conjecture: $\gamma^X(M(G)) = \gamma^{OLD}(G) + c$, where $G \cong P_n$ or C_n and $X \in \{LD, LTD\}$.

Thank you.