Star Colouring of Regular Graphs Meets Weaving and Line Graphs

Shalu M. A. and **Cyriac Antony**

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This talk is based on . . .

- Shalu M. A. and Cyriac Antony (2022), Star colouring of bounded degree graphs and regular graphs, *Discrete Mathematics*, 345 (6), 112850, DOI: 10.1016/j.disc.2022.112850.
- Shalu M. A. and Cyriac Antony (2023), Star colouring and locally constrained graph homomorphisms. Under Review. Preprint link: https://arxiv.org/abs/2312.00086
- Shalu M. A. and Cyriac Antony (2024), Star colouring of regular graphs meets weaving and line graphs. CALDAM 2024

Table of Contents

Introducton & Overview

Notation

Conventions

By graph, we mean finite, simple and undirected graph.

G *contains* H means G contains H as subgraph

Notations

$V(G) =$ vertex set of G

$E(G) =$ edge set of G

For $S \subseteq V(G)$, $\quad G[S] =$ subgraph of G induced by S

Star colouring is a variant of graph colouring.

Star colouring is a variant of graph colouring.

It is used as a model for compression of symmetric sparse matrices (used in computing sparse derivative matrices).

Survey: **What Color Is Your Jacobian? Graph Coloring for Computing Derivatives**, Gebremedhin et al., SIAM Review, (2005).

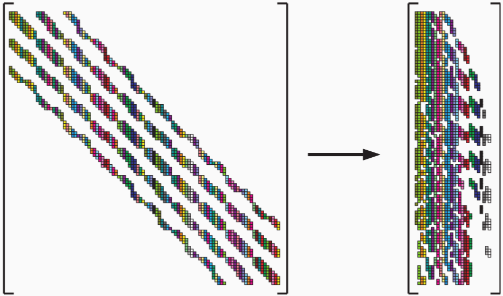


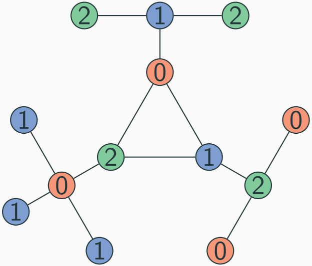
Image credit: (Gebremedhin et al., 2005)

Definition

A *colouring* of a graph G is a function $f\colon V(G) \to \mathbb{Z}$ such that

$$uv \text{ is an edge in } G \implies f(u) \neq f(v).$$

(i.e., (i)——(i) is NOT allowed)



A graph with a 3-colouring

If $f(v) = i$, we say that v is coloured i (by f), and draw v as (i).

A *colouring* of a graph G is a function $f\colon V(G) \to \mathbb{Z}$ such that

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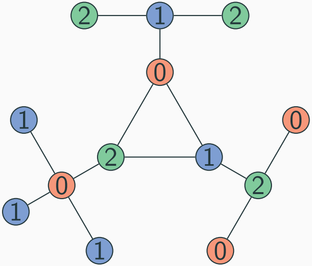

A graph with a 3-colouring

If $f(v) = i$, we say that v is coloured i (by f), and draw v as i.

k-colouring = colouring with at most k colours, say 0,. . . ,k-1

Star Colouring

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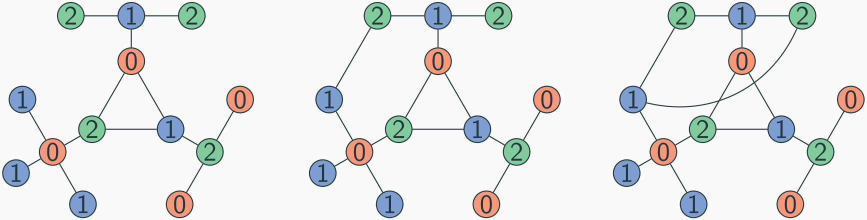
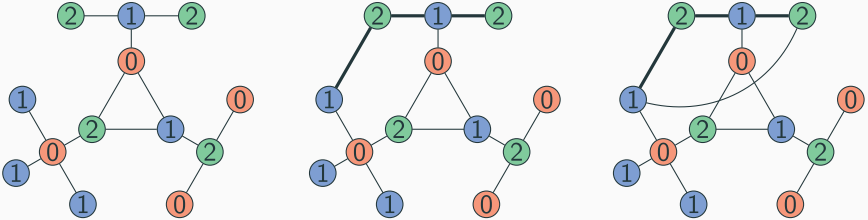


G with a 3-colouring f H with a 3-colouring f J with a 3-colouring f

supergraph ⟶

Star Colouring **Definition**

Star colouring = colouring without i — j — i — j.



G with a 3-colouring f H with a 3-colouring f J with a 3-colouring f

$\xrightarrow{\textit{supergraph}}$

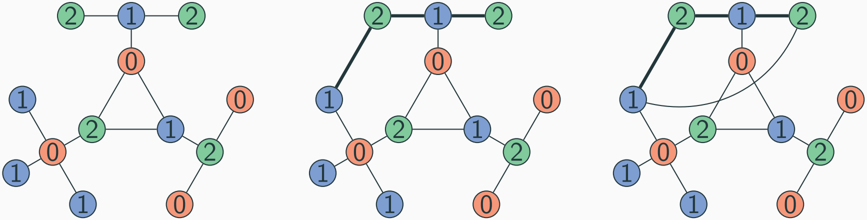
f is a star colouring of G.

f is NOT a star colouring of H.

f is NOT a star colouring of J.

Star colouring = colouring without 

.



G with a 3-colouring f H with a 3-colouring f J with a 3-colouring f

f is a 3-star colouring of G.

f is NOT a 3-star colouring of H.

f is NOT a 3-star colouring of J.

- Characterisation in terms of graph orientations
  (Albertson et al., 2004; Nešetřil and Mendez, 2003)
- Every planar graph is 20-star colourable
  (Albertson et al., 2004).
- Testing 3-star colourability is NP-complete for planar bipartite graphs and line graphs of subcubic graphs
  (Albertson et al., 2004; Lei et al., 2018)
- The minimum #colours required to star colour is polynomial-time computable for
  - P_4-free graphs (Lyons, 2011),
  - P_4-sparse graphs (Yue, 2016), and
  - line graphs of trees (Omoomi et al., 2021)

- If G is a hypercube,
  #colours required $\leq d + 1$ (Fertin et al., 2004)
- If G is 3-regular (i.e., $d = 3$),
  #colours required ≤ 6 (Chen et al., 2013)
  #colours required ≥ 4 (Xie et al., 2014)
- #colours required $\geq \lceil (d + 3)/2 \rceil$ (Fertin et al., 2003)

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Characterise (regular) graphs attaining this bound.

Recognition $\in$ NPC

Given $d \geq 3$ and a d-regular graph G as input, it is NP-complete to test whether G is $\lceil (d+4)/2 \rceil$-star colourable (even when $d = 4$).

$2p$-Regular $(p+2)$-Star Colourable Graphs $\quad p \geq 2$

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Given $p \geq 2$ and a $2p$-regular graph G as input, it is NP-complete to test whether G is $(p+2)$-star colourable (even when $p = 2$).

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Properties

- (diamond, K_4)-free
- $K_{1,p+1}$-free $\implies$ -2 and $p-2$ are eigenvalues of adj. matrix

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Characterisation in terms of

- Graph Orientations
- Graph Homomorphisms
- Edge Decompositions

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For 3-regular graph G,
$L(G)$ is 4-star col. $\iff$ G is bipartite and distance-two 4-col.

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Properties

- (diamond, K_4)-free
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Characterisation in terms of

- Graph Orientations (in-orientations)
- Graph Homomorphisms (locally constrained)
- Edge Decompositions (with weaving pattern)

For 3-regular graph G,
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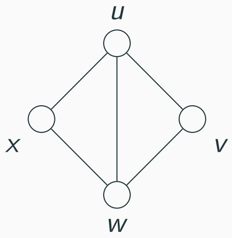
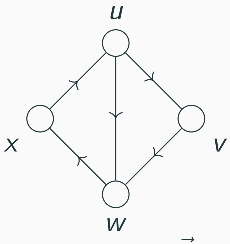
Star Colouring & Orientations

To get an *orientation* of a graph G,
assign some direction on each edge of G.



graph G



orientation $\vec{G}$

$$V(\vec{G}) = V(G)$$

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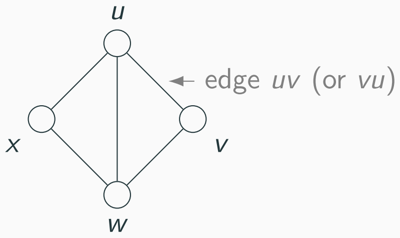

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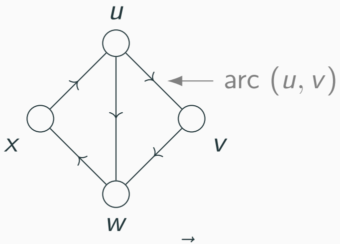

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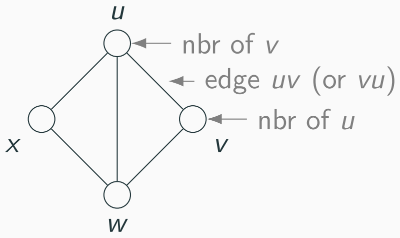

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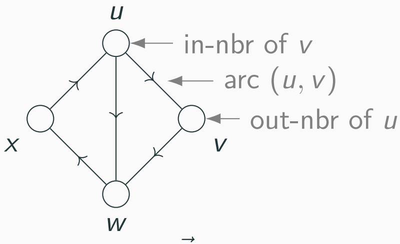

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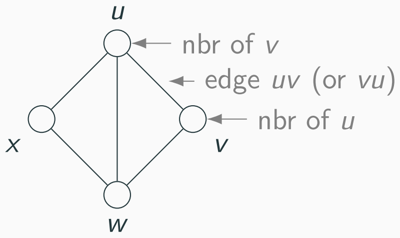

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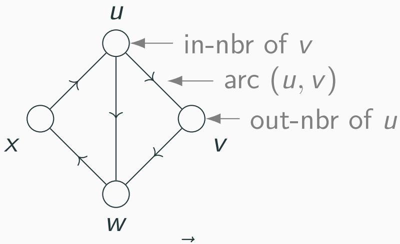

orientation $\vec{G}$

Neighbourhood of v in G
$N(v)$ = set of all nbrs of v (in G)

Out-neighbourhood of v in $\vec{G}$
$N^+(v)$ = set of all out-nbrs of v (in $\vec{G}$)

To get an *orientation* of a graph G,
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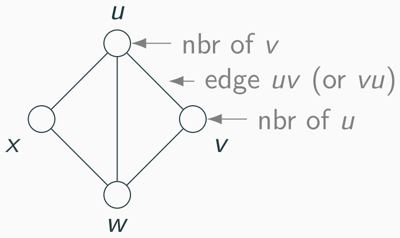

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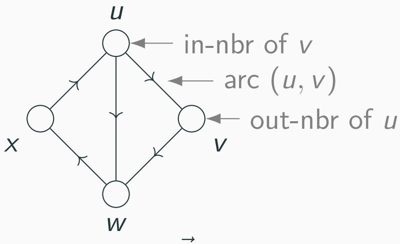
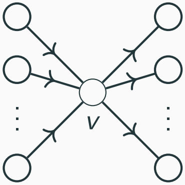


orientation $\vec{G}$

$\vec{G}$ is an *Eulerian orientation* if
#in-nbrs(v)=#out-nbrs(v) for
every vertex v of $\vec{G}$.

An orientation $\vec{G}$ of G is a **in-orientation** of G if there exists a colouring f of $\vec{G}$ such that the following hold for each vertex v of $\vec{G}$:



An orientation $\vec{G}$ of G is a **in-orientation** of G if there exists a colouring f of $\vec{G}$ such that the following hold for each vertex v of $\vec{G}$:

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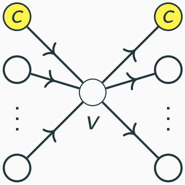

NOT allowed

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(i) no in-nbr and out-nbr of v have the same colour, and
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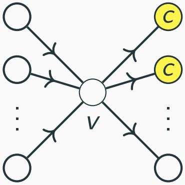

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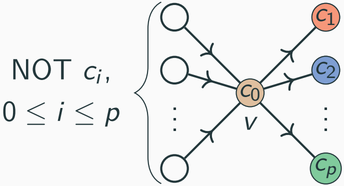

An orientation $\vec{G}$ of G is a ***k*-in-orientation** of G if there exists a k-colouring f of $\vec{G}$ such that the following hold for each vertex v of $\vec{G}$:

(i) no in-nbr and out-nbr of v have the same colour, and
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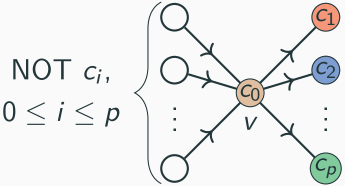

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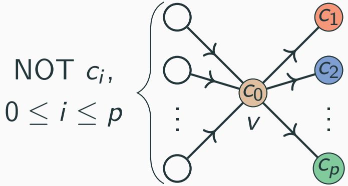

G admits a k-star colouring $\iff$ G admits a k-in-orientation (Albertson et al., 2004; Nešetřil and Mendez, 2003)

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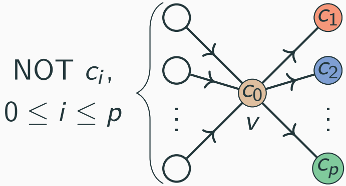
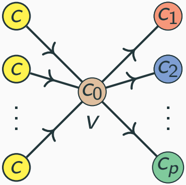
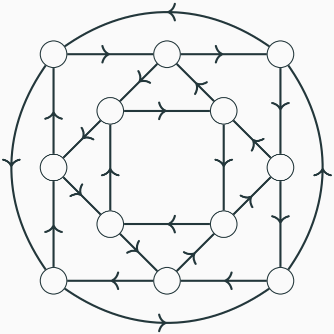


An orientation $\vec{G}$ of G is a ***k*-MINI-orientation** of G if there exists a k-colouring f of $\vec{G}$ such that the following hold for each vertex v of $\vec{G}$:

(i) no in-nbr and out-nbr of v have the same colour
(ii) no two out-nbrs of v have the same colour
(iii) all in-nbrs of v have the same colour



(MINI-orientation = Monochromatic In-Neighbourhood In-orientation)



For each vertex v,

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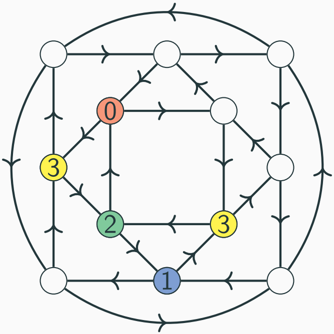
(iii) all in-nbrs of v have the same colour.



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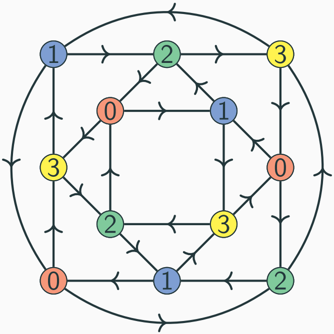

A 4-MINI-orientation of $L(Q_3)$
(with the colouring f).

For each vertex v,

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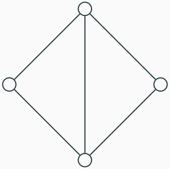
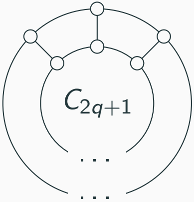
(ii) no two out-nbrs of v have the same colour, and
(iii) all in-nbrs of v have the same colour.

Theorem 1

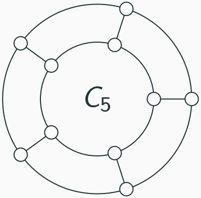
For a $2p$-regular graph G with $p \geq 2$,
G is $(p+2)$-star colourable $\iff$ G has a $(p+2)$-MINI-orientation.



diamond



CL_{2q+1}



e.g.: CL_5

Theorem 2

Diamond and circular ladder graph CL_{2q+1} do not admit a k-MINI-orientation for any $k \in \mathbb{N}$.

Corollary 3

If a 2p-regular G (with $p \geq 2$) contains diamond or CL_{2q+1}, then G is not $(p+2)$-star colourable.

Star Col. & Edge Decompositions

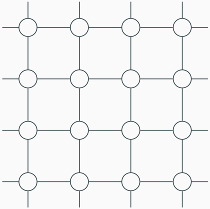
Plain Weaving

Plain weaving in graph theory $\approx$ Alternating projection of link in knot theory

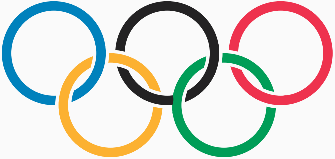
(Akleman et al., 2015)

Plain weave pattern

(Image credit: Adanur, 2020)



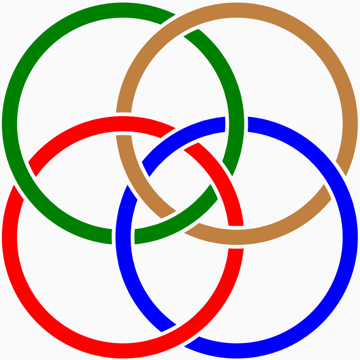
Underlying graph

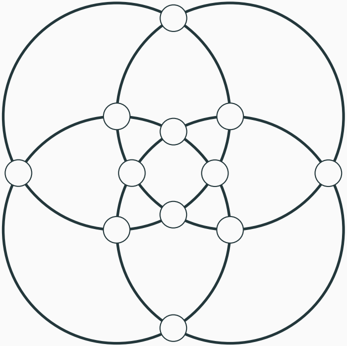


(Image Credits: Github page of spath3 Ti*k*Z library)

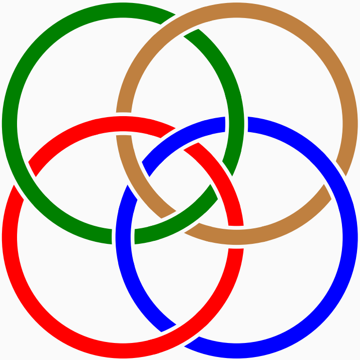
An *plain weaving* of an edge decomposition $\{R_0, R_1, \ldots, R_{q-1}\}$ of a graph G is a function $f\colon V(G) \to \mathbb{Z}_q$ such that for each $i \in \mathbb{Z}_q$ and each edge $uv \in R_i$, either $f(u) = i$ or $f(v) = i$.

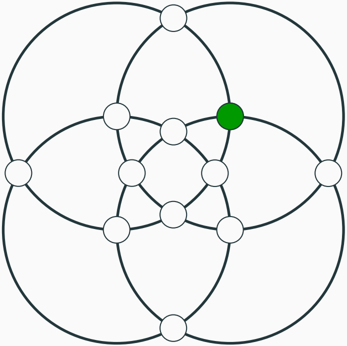
Plain weaving, colouring & orientation



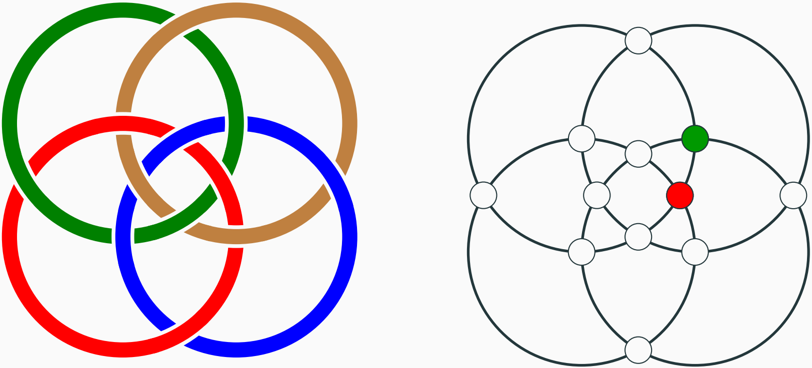


Plain weaving, colouring & orientation





Plain weaving, colouring & orientation



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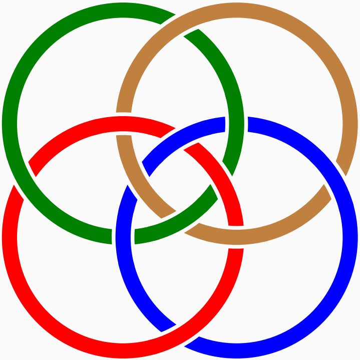
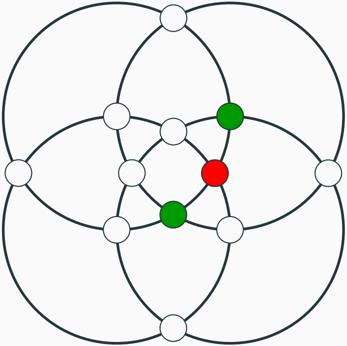
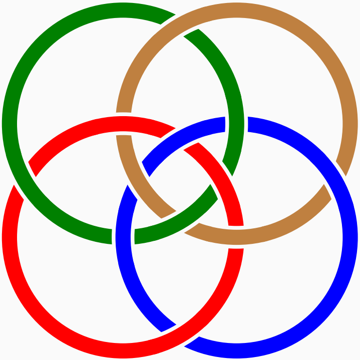
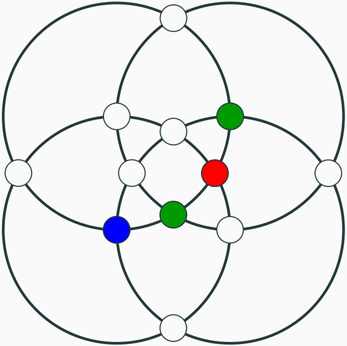




Plain weaving, colouring & orientation





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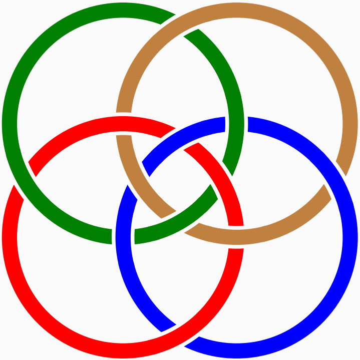
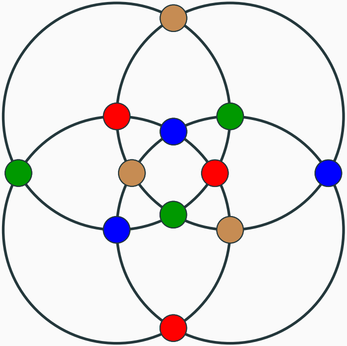




Theorem 4

Let $p \geq 2$ and $q \geq 2$. Let G be a $2p$-regular graph, and let $\vec{G}$ be an orientation of G. Then, $\vec{G}$ is an Eulerian q-MINI-orientationj of G if and only if G admits an edge decomposition $S = \{H_0, H_1, \ldots, H_{q-1}\}$ that satisfies the following:
(i) each H_i is p-regular ($i \in \mathbb{Z}_q$);
(ii) orientation induced by S is $\vec{G}$; and
(iii) for distinct $i, j \in \mathbb{Z}_q$ and distinct $u, v \in V(H_i) \cap V(H_j)$,
$uv \notin E(G)$ *and* $N_G(u) \cap N_G(v) = N^+_{\vec{G}}(u) \cap N^+_{\vec{G}}(v)$.

Plain weaving & colouring

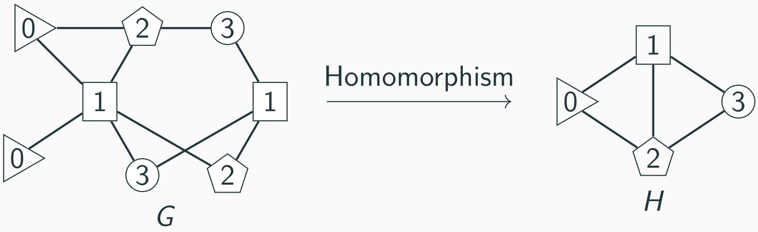
Theorem 5

Let G be a $2p$-regular graph with $p \geq 2$. Then, G admits a $(p+2)$-star colouring if and only if G an edge decomposition $S = \{H_0, H_1, \ldots, H_{p+1}\}$ such that the following hold:
(i) each H_i is p-regular ($i \in \mathbb{Z}_{p+2}$)
(let us call orientation induced by S as $\vec{G}$); and
(ii) for distinct $i, j \in \mathbb{Z}_{p+2}$ and distinct $u, v \in V(H_i) \cap V(H_j)$,
$uv \notin E(G)$ and $N_G(u) \cap N_G(v) = N^+_{\vec{G}}(u) \cap N^+_{\vec{G}}(v)$. □

Star Colouring & Homomorphisms

Let G and H be graphs.

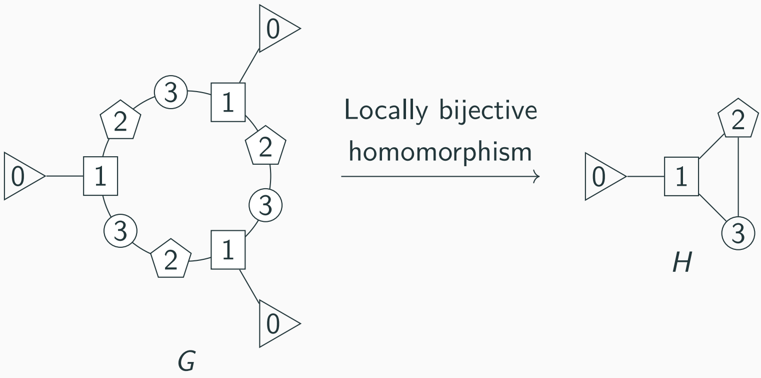
A homomorphism from G to H is a function $\psi\colon V(G) \to V(H)$ s. t. uv is an edge in $G \implies \psi(u)\psi(v)$ is an edge in H.



ψ maps triangles to triangle, circles to circle, and so on.

A **Locally Bijective Homomorphism (LBH)** from G to H is a function $\psi\colon V(G) \to V(H)$ such that for each vertex v of G, ψ maps neighbourhood $N_G(v)$ bijectively to $N_H(\psi(v))$.

Notation: $G \xrightarrow{\text{LBH}} H$ (here, ψ maps triangles to triangle, ...)



Areas Related to LBH

- Topology
- Algebra
- Combinatorics
- Geometry

Surveys:
(Fiala and Kratochvíl, 2008) Locally constrained graph homomorphisms – structure, complexity, and applications

(Fiala et al., 2008) Locally constrained graph homomorphisms and equitable partitions

Theorem 6 (see Fiala and Kratochvíl, 2008)

$G \xrightarrow{LBH} H \quad \Longrightarrow \quad \mathit{char}(H;x) \text{ divides } \mathit{char}(G;x).$

Theorem 7 (Dvořák et al., 2013)

For a 3-regular graph G,
$L(G)$ is 4-star colourable $\iff G \xrightarrow{LBH} Q_3$

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Theorem 8

For a 3-regular graph G,

$L(G)$ *is 4-star colourable* $\iff$ G *is bipartite and distance-two 4-colourable*

(distance-two 4-colouring = 4-colouring without 

)

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(distance-two 4-colouring = 4-colouring without i — j — i)

Corollary 9

It is NP-complete to test whether a planar 4-regular graph is 4-star colourable.

Theorem 7 [Dvořák et al., 2013] (Restated)

For a 3-regular graph H,
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Rephrasal of Theorem 7:

Theorem 10 (Dvořák et al., 2013)

For a 3-regular graph H,
$L(H)$ *is 4-star colourable* $\iff L(H) \xrightarrow{LBH} L(Q_3)$

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G is 4-star colourable $\iff G \xrightarrow{\text{LBH}} L(Q_3)$.

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Claim: For every $K_{1,3}$-free 4-regular graph G,
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We define a sequence of graphs $G_4, G_6, G_8 \ldots$, where $G_4 \cong L(Q_3)$.

Theorem 11

For a $K_{1,p+1}$-free $2p$-regular graph G with $p \geq 2$,
G is $(p+2)$-star colourable $\iff G \xrightarrow{LBH} G_{2p}$.

Star Colouring & Line Graphs

Motivation

For a 3-regular graph H,
$L(H)$ is 4-star colourable $\iff L(H) \xrightarrow{\text{LBH}} L(Q_3)$

For every $K_{1,3}$-free 4-regular graph G,
G is 4-star colourable $\implies G$ is a line graph

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For a $K_{1,p+1}$-free $2p$-regular graph G,
does G $(p+2)$-star colourable graphs $\implies$ G is a line graph?

Are $G_4, G_6, G_8 \ldots$ line graphs?

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For a $K_{1,p+1}$-free $2p$-regular graph G,
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Are $G_4, G_6, G_8 \ldots$ line graphs?
No, except for G_4.

Motivation

G is $2p$-regular & $(p+2)$-star colourable	$\Longrightarrow$	G is (diamond, K_4)-free. $(p+1)(p+2)$ divides $\lvert V(G)\rvert$.
& in addition, G is $K_{1,p+1}$-free	$\Longrightarrow$	
& in addition, $\lvert V(G)\rvert = (p+1)(p+2)$	$\Longrightarrow$	$G \cong G_{2p}$

Motivation

G is $2p$-regular & $(p+2)$-star colourable	$\Longrightarrow$	G is (diamond, K_4)-free. $(p+1)(p+2)$ divides $\lvert V(G)\rvert$.
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Clique graph of H = Interesection graph of maximal cliques in H.
G is clique graph means that G is the clique graph of some graph.

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& in addition, $\lvert V(G)\rvert = (p+1)(p+2)$	$\Longrightarrow$	$G \cong G_{2p} \cong L^*(K_{p+2})$.

Clique graph of H = Interesection graph of maximal cliques in H.
G is clique graph means that G is the clique graph of some graph.

$L^*(H)$ = underlying undirected graph of line digraph of H.

What is a line digraph?

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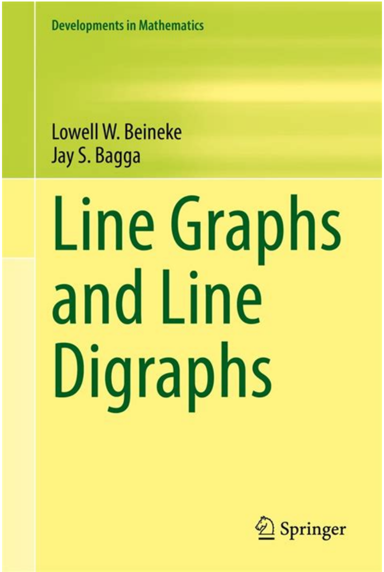

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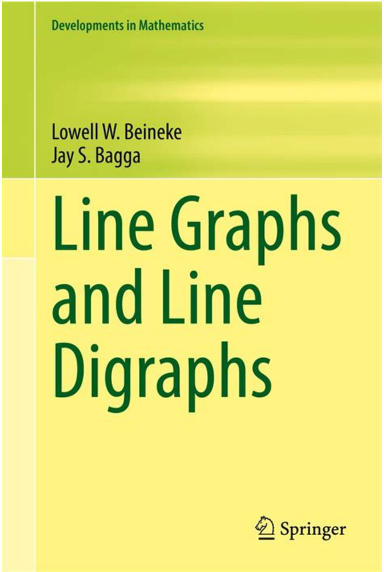


Discrete Mathematics Letters
www.dmlett.com

Review Article

A survey of line digraphs and generalizations*

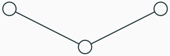
Jay S. Bagga[1,†], Lowell W. Beineke[2]

Discrete Math. Lett. **6** (2021) 68–83
DOI: 10.47443/dml.2021.s109

Line Digraph

Definition

Line graph:



in G

in $L(G)$ on top of G

in $L(G)$

Line Digraph

Definition

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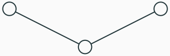
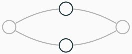

in G


in $L(G)$ on top of G

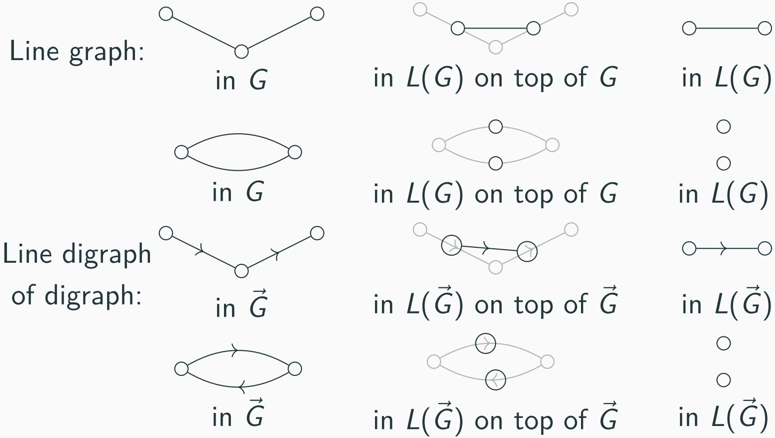

in $L(G)$


in G


in $L(G)$ on top of G


in $L(G)$

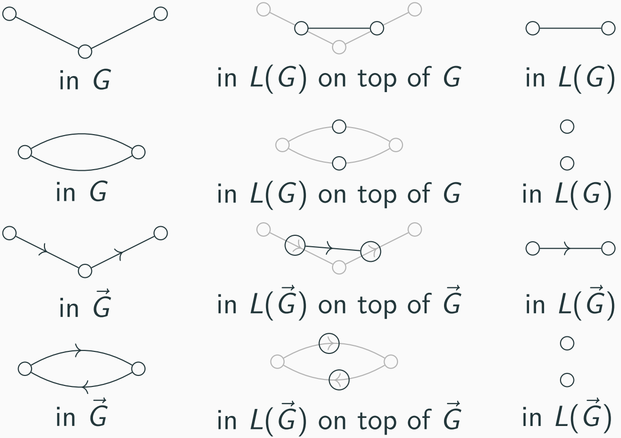
Line Digraph



Line Digraph — Definition

Line graph:

Line digraph of digraph:



To get the **line digraph of a graph** G,
replace each edge of G by  (to get $\vec{G}$)
& then perform line digraph operation (on $\vec{G}$).

For an (undirected) graph H,
the **line digraph** $\vec{L}(H)$ of H is the oriented graph with
Vertex set $= \{(u, v), (v, u) : u, v \in V(H) \text{ and } uv \in E(H)\}$
Arcs: $(u, v) \to (v, w)$ for $u, v, w \in V(H)$, provided $u \neq w$

(Bagga and Beineke, 2021).

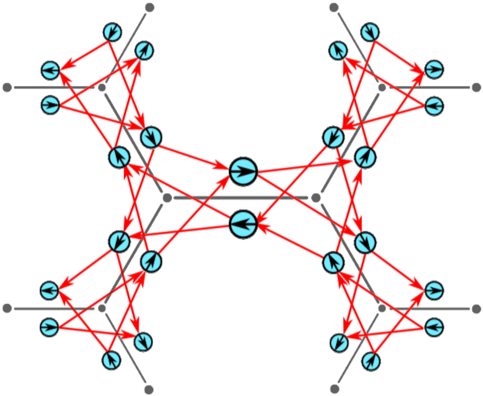
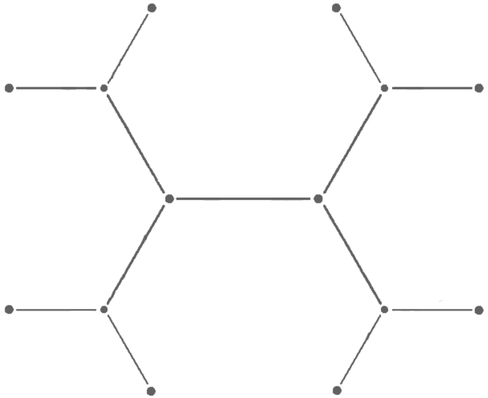


Image credit: (Parzanchevski, 2020).

Star Colouring and Line Graph



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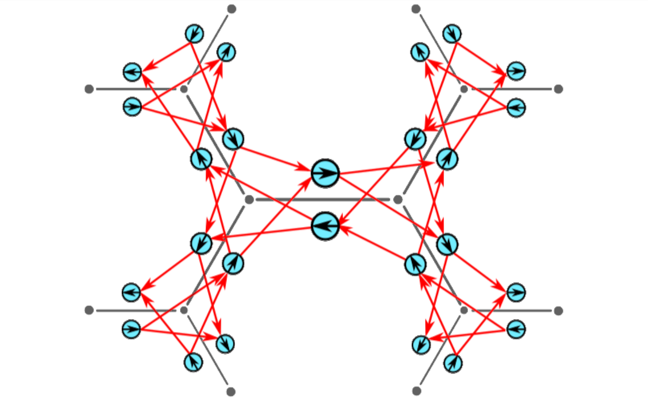

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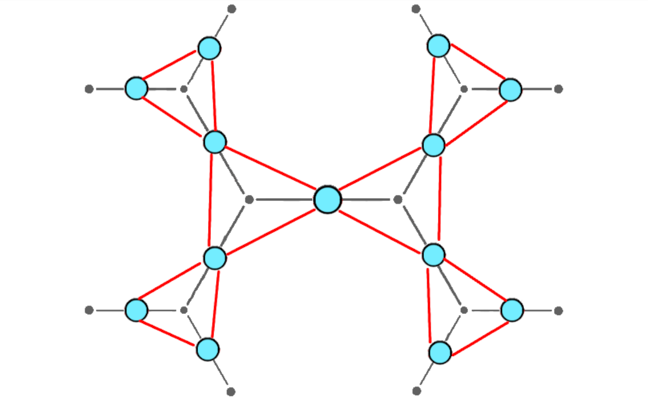

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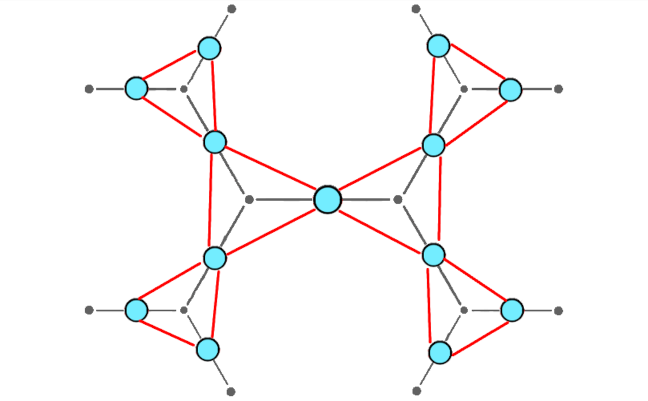

Theorem 12

For every graph H, $L^*(H) \xrightarrow{LBH} L(H)$.

Star Colouring and Line Graph

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$$\text{char}(L(K_{p+2}); x) = (x-2p)(x-p+2)^{p+1}(x+2)^{(p-1)(p+2)/2}$$

(Beineke and Bagga, 2021)

Corollary 14

For a $K_{1,p+1}$-free $2p$-regular graph G with $p \geq 2$,
$G \xrightarrow{LBH} L(K_{p+2})$ *and thus -2 and $p-2$ are eigenvalues of G.*

Future Directions

Ongoing Work

1. Determine spectra of $L^*(H)$.
2. Characterise constrained homorphisms related to star colouring in terms of edge decompositions.

Future Directions

1. Charatcerise $(2p+1)$-regular $(p+2)$-star colourable graphs.
2. Characterise graphs that do not admit MINI-orientaton (similar to diamond).
3. Use weaving to study 1-cover or 2-covers of matchings and even-degree graphs.

References

Adanur, S. (2020). ***Handbook of weaving.*** Boca Raton: CRC press.

Akleman, E., J. Chen, and J. L. Gross (2015). **"Extended graph rotation systems as a model for cyclic weaving on orientable surfaces".** In: *Discrete Applied Mathematics* 193, pp. 61–79.

Albertson, M. O., G. G. Chappell, H. A. Kierstead, A. Kündgen, and R. Ramamurthi (2004). **"Coloring with no 2-colored P_4's".** In: *The Electronic Journal of Combinatorics* 11.1, p. 26.

Bagga, J. S. and L. W. Beineke (2021). **"A survey of line digraphs and generalizations".** In: *DML. Discrete Mathematics Letters* 6, pp. 68–83.

Beineke, L. W. and J. S. Bagga (2021). ***Line graphs and line digraphs.*** Vol. 68. Developments in Mathematics. Cham: Springer.

Chen, M., A. Raspaud, and W. Wang (2013). **"6-star-coloring of subcubic graphs".** In: *Journal of Graph Theory* 72.1-2, pp. 128–145.

References (cont'd)

- Dvořák, Z., B. Mohar, and R. Šámal (2013). **"Star chromatic index".** In: *Journal of Graph Theory* 72.3-4, pp. 313–326.
- Fertin, G., E. Godard, and A. Raspaud (2003). **"Acyclic and k-distance coloring of the grid".** In: *Information Processing Letters* 87.1, pp. 51–58.
- Fertin, G., A. Raspaud, and B. Reed (2004). **"Star coloring of graphs".** In: *Journal of Graph Theory* 47.3, pp. 163–182.
- Fiala, J. and J. Kratochvíl (2008). **"Locally constrained graph homomorphisms – structure, complexity, and applications".** In: *Computer Science Review* 2.2, pp. 97–111.
- Fiala, J., D. Paulusma, and J. A. Telle (2008). **"Locally constrained graph homomorphisms and equitable partitions".** In: *European Journal of Combinatorics* 29.4, pp. 850–880. URL: dro.dur.ac.uk/7423/1/7423.pdf.

Lei, H., Y. Shi, and Z.-X. Song (2018). **"Star chromatic index of subcubic multigraphs".** In: *Journal of Graph Theory* 88.4, pp. 566–576.

Lyons, A. (2011). **"Acyclic and star colorings of cographs".** In: *Discrete Applied Mathematics* 159.16, pp. 1842–1850.

Nešetřil, J. and P. O. de Mendez (2003). **"Colorings and homomorphisms of minor closed classes".** In: *Discrete and Computational Geometry*. Springer, pp. 651–664.

Omoomi, B., E. Roshanbin, and M. V. Dastjerdi (2021). **"A polynomial time algorithm to find the star chromatic index of trees".** In: *The Electronic Journal of Combinatorics* 28.1, p1.6, 16.

Parzanchevski, O. (2020). **"Ramanujan graphs and digraphs".** In: *Analysis and geometry on graphs and manifolds. July 31 – August 4, 2017*. Cambridge: Cambridge University Press, pp. 344–367.

Xie, D., H. Xiao, and Z. Zhao (2014). **"Star coloring of cubic graphs".** In: *Information Processing Letters* 114.12, pp. 689–691.

Yue, J. (2016). **"Acyclic and star coloring of P_4-reducible and P_4-sparse graphs".** In: *Applied Mathematics and Computation* 273, pp. 68–73.

Thank you

Questions?